

STRONG MAXIMUM PRINCIPLE FOR MEAN CURVATURE OPERATORS ON SUBRIEMANNIAN MANIFOLDS

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ABSTRACT. We study the strong maximum principle for horizontal (p -) mean curvature operator and p -(sub)laplacian operator on subriemannian manifolds including, in particular, Heisenberg groups and Heisenberg cylinders. Under a certain Hormander type condition on vector fields, we show the strong maximum principle holds in higher dimensions for two cases: (a) the touching point is nonsingular; (b) the touching point is an isolated singular point for one of comparison functions. For a background subriemannian manifold with local symmetry of isometric translations, we have the strong maximum principle for associated graphs which include, among others, intrinsic graphs with constant horizontal (p -) mean curvature. As applications, we show a rigidity result of horizontal (p -) minimal hypersurfaces in any higher dimensional Heisenberg cylinder and a pseudo-halfspace theorem for any Heisenberg group.

1. Introduction and statement of the results

E. Hopf probably is the first person who studied the strong maximum principle (SMP in short) of elliptic operators in the generality. See his paper [21] of 1927 or Theorem 3.5 in [20]. For earlier results, under more restrictive hypotheses, see references in [32]. This principle has been extended to the case for certain quasilinear elliptic operators of second order ([20]). In 1969 J.-M. Bony ([5]) studied, among others, the SMP for linear operators of Hörmander type including some known subelliptic operators. Bony's SMP has been applied to study various geometric problems. See, for instance, B. Andrews' work on noncollapsing in mean-convex mean curvature flow ([2]) or S. Brendle's solution to the Lawson conjecture ([6]). In Subsection A of the Appendix, we give a brief review of Bony's SMP.

In this paper we first extend Bony's SMP to the quasilinear case and then apply it to (generalized) mean curvature operators in subriemannian geometry, including p -sublaplacian and usual horizontal (p -) mean curvature. We consider quasilinear operators Q of second order:

$$(1.1) \quad Q\phi = a^{ij}(x, D\phi)D_{ij}\phi + b(x, \phi, D\phi)$$

where $x = (x^1, \dots, x^{m+1})$ is contained in a domain Ω of R^{m+1} , $m \geq 1$. The coefficients $a^{ij}(x, p)$ ($b(x, z, p)$, resp.) of Q are assumed to be defined and C^∞ smooth (for simplicity) for all values of (x, p) ((x, z, p) , resp.) in the set $\Omega \times R^{m+1}$ ($\Omega \times R \times R^{m+1}$, resp.). Let $\mathcal{L}(X_1, \dots, X_r)$ denote the smallest C^∞ -module which contains C^∞ vector fields X_1, \dots, X_r on Ω and is closed under the Lie bracket (see (7.4) in the

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Appendix for precise definition). The following comparison principle is a straightforward application of Bony's SMP (Theorem 3.1 in [5] or see Theorem A7 in the Appendix).

Theorem A. *Let $\phi, \psi \in C^\infty(\Omega)$ satisfy $Q\phi \geq Q\psi$ in Ω . Assume*

(1) *(a^{ij}) is nonnegative and $a^{ij} = a^{ji}$;*

(2) *$\frac{\partial b}{\partial z} \leq 0$;*

(3) *there exist vector fields X_1, \dots, X_r and Y of class C^∞ (depending on $D\phi(x)$) such that*

$$a^{ij}(x, D\phi(x))D_{ij} = \sum_{k=1}^r X_k^2 + Y.$$

Let Γ be an integral curve of a vector field $Z \in \mathcal{L}(X_1, \dots, X_r)$. Suppose $\phi - \psi$ achieves a nonnegative maximum in Ω at a point of Γ . Then the maximum is attained at all points of Γ .

Let e_1, \dots, e_{m+1} be independent C^∞ vector fields on Ω . Consider second order quasilinear operators Q' of the form:

$$(1.2) \quad Q'\phi = a^{ij}(x, e_1\phi, \dots, e_{m+1}\phi)e_ie_j\phi + b(x, \phi, e_1\phi, \dots, e_{m+1}\phi)$$

where $x = (x^1, \dots, x^{m+1})$ is contained in a domain Ω of R^{m+1} , $m \geq 1$. We have the following moving frame version of Theorem A.

Theorem A'. *Let $\phi, \psi \in C^\infty(\Omega)$ satisfy $Q'\phi \geq Q'\psi$ in Ω . Assume*

(1) *(a^{ij}) is nonnegative and $a^{ij} = a^{ji}$;*

(2) *$\frac{\partial b}{\partial z} \leq 0$;*

(3) *there exist vector fields X_1, \dots, X_r and Y of class C^∞ (depending on $e_1\phi(x), \dots, e_{m+1}\phi(x)$) such that*

$$a^{ij}(x, e_1\phi(x), \dots, e_{m+1}\phi(x))e_ie_j = \sum_{k=1}^r X_k^2 + Y.$$

Let Γ be an integral curve of a vector field $Z \in \mathcal{L}(X_1, \dots, X_r)$. Suppose $\phi - \psi$ achieves a nonnegative maximum in Ω at a point of Γ . Then the maximum is attained at all points of Γ .

Let

$$\begin{aligned} \tilde{a}^{ij}(x) &: = a^{ij}(x, D\phi(x)) \text{ in Theorem A} \\ (&: = a^{ij}(x, e_1\phi(x), \dots, e_{m+1}\phi(x)) \text{ in Theorem A', resp.} \end{aligned}$$

In practice, condition (3) in Theorem A (Theorem A', resp.) can be replaced by

$$(1.3) \quad \text{rank}(\tilde{a}^{ij}(x)) = \text{constant } \tilde{r}$$

Theorem \tilde{A} . *Theorem A (Theorem A', resp.) holds if condition (3) is replaced by constant rank condition (1.3).*

In applications, we usually assume $\phi \leq \psi$ and $\phi = \psi$ at a point p_0 . Then we conclude $\phi \equiv \psi$ on a hypersurface Σ containing p_0 if the Lie span $\mathcal{L}(X_1, \dots, X_r) = C^\infty(\Sigma, T\Sigma)$. We are going to apply Theorem \tilde{A} to generalized mean curvature or p -laplacian $H_{\phi,p}$ with $p \geq 0$.

A subriemannian manifold is a (C^∞) smooth manifold M equipped with a non-negative inner product $\langle \cdot, \cdot \rangle^*$ on T^*M , its cotangent bundle, i.e., $\langle v, v \rangle^* \geq 0$

0 for any cotangent vector v . When $\langle \cdot, \cdot \rangle^*$ is positive definite, the definition is equivalent to the usual definition of Riemannian manifold using positive definite inner product on tangent bundle TM . However for a degenerate $\langle \cdot, \cdot \rangle^*$, it is difficult to define on the whole TM instead of T^*M . So the above definition using T^*M of subriemannian manifold generalizes the notion of Riemannian manifold in a unified way.

Let ϕ be a (C^∞ smooth, say) defining function (i.e., $d\phi \neq 0$) on a subriemannian manifold $(M, \langle \cdot, \cdot \rangle^*)$ of dimension $m+1$ with $|d\phi|_* := (\langle d\phi, d\phi \rangle^*)^{1/2} \neq 0$ (note that at a point where $\langle \cdot, \cdot \rangle^*$ is degenerate, we may have $d\phi \neq 0$ while $|d\phi|_* = 0$). Let dv_M be a background volume form on M , i.e., a given $(m+1)$ -form which is nowhere vanishing. For M being Riemannian, we may take the associated volume form as a background volume form. For degenerate M , a background volume form is a choice independent of the (degenerate) metric $\langle \cdot, \cdot \rangle^*$. With respect to $\langle \cdot, \cdot \rangle^*$ we can then talk about the interior product of a 1-form ω with dv_M : $\omega \lrcorner dv_M$ being an m -form such that $\eta \wedge (\omega \lrcorner dv_M) = \langle \eta, \omega \rangle^* dv_M$ for any 1-form η .

We define a function $H_{\phi,p}$ on M with $p \geq 0$ by the following formula:

$$(1.4) \quad d\left(\frac{d\phi}{|d\phi|_*^{1-p}} \lrcorner dv_M\right) = H_{\phi,p} dv_M.$$

For $p = 0$, $H_{\phi,p}$, denoted as H_ϕ often, is called (Riemannian, subriemannian, or horizontal) mean curvature while, for $p > 0$, $H_{\phi,p}$ is so called p -laplacian or p -sublaplacian. For the variational formulation, consider the following energy functional:

$$\mathcal{F}_p(\phi) := \int_{\Omega} \left(\frac{1}{p} |d\phi|_*^p + H\phi\right) dv_M$$

(H being prescribed subriemannian mean curvature or p -sublaplacian) where $\Omega \subset M$ is a smooth bounded domain. Let $\phi_\varepsilon = \phi + \varepsilon\rho$ where $\rho \in C_0^\infty(\Omega)$. Compute the first variation of \mathcal{F}_p : (omitting the volume form dv_M)

$$(1.5) \quad \begin{aligned} & \frac{d\mathcal{F}_p(\phi_\varepsilon)}{d\varepsilon} \Big|_{\varepsilon=0\pm} \\ &= c_p \int_{S(\phi)} |d\rho|_*^p + \int_{\Omega \setminus S(\phi)} |d\phi|_*^{p-2} \langle d\phi, d\rho \rangle^* + \int_{\Omega} H\rho \end{aligned}$$

where $c_p = \pm 1$ for $p = 1$, $c_p = 0$ for $1 < p < \infty$, and $S(\phi)$ is the set where $|d\phi|_* = 0$, called the singular set of ϕ (cf. (1.4) in [14]). From (1.5) we learn that for $p = 1$, the first term involving the singular set $S(\phi)$ is not negligible. So in the proof of the maximum principle (or comparison theorem), we need to worry about the size of $S(\phi)$ (see [12], [14] for more details). In Lemma 5.2 of this paper, we extend the maximum principle (comparison theorem) to general subriemannian manifolds. This is necessary in order to show the SMP near singular points (where $|d\phi|_* = 0$). In this paper we mainly deal with the SMP near nonsingular points (where $|d\phi|_* \neq 0$, $|d\psi|_* \neq 0$). For the SMP near singular points, we only discuss the situation that the reference singular point is isolated for at least one comparison hypersurface. In general, the problem of the SMP near singular points is still open.

Let ϕ and ψ be defining functions for hypersurfaces Σ_1 and Σ_2 in a subriemannian manifold $(M, \langle \cdot, \cdot \rangle^*)$ of dimension $m+1$, resp. ($m \geq 1$). I.e., Σ_1 (Σ_2 , resp.) is defined by $\phi = 0$ ($\psi = 0$, resp.). Suppose Σ_1 and Σ_2 are tangent to each other at a point p_0 where $|d\phi|_* \neq 0$, $|d\psi|_* \neq 0$. Define $G : T^*M \rightarrow TM$ by $\omega(G(\eta)) =$

$\langle \omega, \eta \rangle^*$ for $\omega, \eta \in T^*M$. Let $\xi := \text{Range}(G)$. Throughout this paper we assume

$$(1.6) \quad \dim \xi = \text{constant } m + 1 - l$$

near p_0 with l being a nonnegative integer unless stated otherwise. We call l the degree of degeneracy of M . Note that $\dim \ker G = l$. The following rank condition:

$$(1.7) \quad \text{rank}(\mathcal{L}(X_1, \dots, X_{m-l})) = m$$

for any local sections X_1, \dots, X_{m-l} of ξ , which are independent wherever defined, is important. It means that any $(m-l)$ -dimensional subspace of local sections of ξ can generate m -dimensional spaces. Let

$$\mathcal{L}(\xi \cap T\Sigma_1) = \mathcal{L}(X_1, \dots, X_{m-l})$$

where X_1, \dots, X_{m-l} form a basis of local sections of $\xi \cap T\Sigma_1$ near p_0 . Similarly we can define

$$\mathcal{L}(\xi) = \mathcal{L}(X_1, \dots, X_{m-l}, X_{m+1-l})$$

where $X_1, \dots, X_{m-l}, X_{m+1-l}$ form a basis of local sections of ξ near p_0 . Note that both $\mathcal{L}(\xi \cap T\Sigma_1)$ and $\mathcal{L}(\xi)$ are independent of choice of a basis of local sections.

Theorem B. *Suppose we are in the situation described above, in particular, $|d\phi|_* \neq 0$, $|d\psi|_* \neq 0$ at p_0 . For $p \geq 0$, Assume*

$$H_{\psi,p} + b(x, \psi, D\psi) \leq H_{\phi,p} + b(x, \phi, D\phi)$$

where b is a C^∞ smooth function for all values of (x, z, \cdot) and satisfies $\frac{\partial b}{\partial z} \leq 0$. Moreover, assume $\psi \geq \phi$ near p_0 , $\psi = \phi = 0$ at p_0 . We have

(a) if we further assume the rank condition (1.7) holds near p_0 , then $\psi = \phi = 0$ on Σ_1 near p_0 . I.e., Σ_2 coincides with Σ_1 near p_0 .

(b) in the case of $p > 0$, if we further assume $\text{rank}(\mathcal{L}(\xi)) = m + 1$, then $\psi = \phi$ near p_0 .

Next we want to show that in a certain situation the assumption in Theorem B can be achieved. Suppose we have a one-parameter family of diffeomorphisms Ψ_a , $a \in (-\delta, \delta)$ for small δ , say, in a small neighborhood U of p_0 in M (where ϕ and ψ are defined), i.e., $\Psi_a = \text{Id}$ for $a = 0$ and $\Psi_{a+b} = \Psi_a \circ \Psi_b$ wherever defined.

Definition 1.1. *Let $(M, \langle \cdot, \cdot \rangle^*, dv_M)$ be a subriemannian manifold with a background volume form dv_M . $(M, \langle \cdot, \cdot \rangle^*, dv_M)$ is said to have isometric translations (Ψ_a) near $p_0 \in M$ if there exists a one-parameter family of local diffeomorphisms Ψ_a , $a \in (-\delta, \delta)$ for small $\delta > 0$, say, in a small neighborhood U of p_0 , such that*

(a) *(preserving $\langle \cdot, \cdot \rangle^*$) $\langle \Psi_a^*(\omega), \Psi_a^*(\eta) \rangle^* = \langle \omega, \eta \rangle^*$ for $a \in (-\delta, \delta)$ and $\omega, \eta \in T^*U$.*

(b) *(preserving dv_M) $\Psi_a^*(dv_M) = dv_M$ for $a \in (-\delta, \delta)$.*

We say the defining function ϕ of a (local) hypersurface Σ passing through p_0 is compatible with $\{\Psi_a\}$ or $\{\Psi_a\}$ is compatible with the defining function ϕ if

$$(1.8) \quad \phi(\Psi_a(x)) = \phi(x) - a$$

for $x \in \Sigma \cap U$ (and hence $x \in U$) and $a \in (-\delta, \delta)$. $\{\Psi_a\}$ is said to be transversal to a hypersurface Σ of U if

$$\frac{d\Psi_a(x)}{da} \Big|_{a=0} \notin T_x \Sigma$$

for all $x \in \Sigma$. We have a notion of (generalized) "hypersurface" mean curvature (p -sublaplacian) $H_{\Sigma,p}$ defined on $\Sigma \setminus S(\phi)$ as follows. At a point where $|d\phi|_* \neq 0$, for $p \geq 0$, we define p -subriemannian area (or volume) element $dv_{\phi,p}$ for the hypersurface $\{\phi = c, \text{ a constant}\}$ by

$$dv_{\phi,p} := \frac{d\phi}{|d\phi|_*^{1-p}} \lrcorner dv_M.$$

Define unit p -normal ν_p to a hypersurface $\Sigma := \{\phi = c\}$ by the formula

$$\nu_p \lrcorner dv_M = dv_{\phi,p}.$$

We can now define "hypersurface" mean curvature $H_{\Sigma,p}$ on Σ through a variational formula:

$$\delta_{f\nu_p} \int_{\Sigma} dv_{\phi,p} = \int_{\Sigma} f H_{\Sigma,p} dv_{\phi,p}$$

for $f \in C_0^\infty(\Sigma \setminus S(\phi))$. In Subsection B of the Appendix, we show that $H_{\phi,p} = H_{\Sigma,p}$ on $\Sigma \setminus S(\phi)$ (see Proposition B.1).

Theorem C. *Suppose $(M, \langle \cdot, \cdot \rangle^*, dv_M)$ of dimension $m+1$ has isometric translations Ψ_a near $p_0 \in M$. Let U be the neighborhood of p_0 in Definition 1.1. Suppose $\{\Psi_a\}$ is transversal to Σ_1 and Σ_2 in U . Choose ϕ and ψ to be defining functions for hypersurfaces Σ_1 and Σ_2 in U , resp. (I.e., Σ_1 (Σ_2 , resp.) is defined by $\phi = 0$ ($\psi = 0$, resp.)), compatible with $\{\Psi_a\}$. Suppose Σ_1 and Σ_2 are tangent to each other at p_0 where $|d\phi|_* \neq 0$, $|d\psi|_* \neq 0$. Assume*

- (1) *For any $q \in \Sigma_1 \cap U$, there exists $\delta > 0$ such that $\Psi_{a(q)}(q) \in \Sigma_2$, and*
- (2) *For some $p \geq 0$, $H_{\Sigma_2,p}(\Psi_{a(q)}(q)) \leq H_{\Sigma_1,p}(q)$ for any $q \in \Sigma_1 \cap U$.*

Moreover, assume the rank condition (1.7) holds near p_0 . Then Σ_2 coincides with Σ_1 near p_0 .

Next we want to give a more analytic description in terms of graphs. We will show the existence of some special coordinates for a subriemannian manifold having local isometric translations as shown below.

Theorem \hat{C} . *Let $(M, \langle \cdot, \cdot \rangle^*, dv_M)$ be an $(m+1)$ -dimensional subriemannian manifold with a background volume form dv_M . Suppose $(M, \langle \cdot, \cdot \rangle^*, dv_M)$ has isometric translations Ψ_a , $a \in (-\delta, \delta)$ for $\delta > 0$, near $p_0 \in M$, transversal to a hypersurface Σ passing through p_0 . Then we can find local coordinates x^1, x^2, \dots, x^{m+1} in a neighborhood V of p_0 such that*

- (1) *Σ is described by $x^{m+1} = 0$ in V , p_0 is the origin, $x^j \circ \Psi_a = x^j$ at $q \in V$ for any a such that $\Psi_a(q) \in V$, $1 \leq j \leq m$, and $x^{m+1} \circ \Psi_a = x^{m+1} + a$;*
- (2) *$A(\Psi_a(q)) = A(q)$ for any $q \in V$ and any a such that $\Psi_a(q) \in V$ if we write $dv_M(q) = A(q) dx^1 \wedge \dots \wedge dx^{m+1}$.*

Call this system of special coordinates above in Theorem \hat{C} (a system of) translation-isometric coordinates.

Definition 1.2. *Take a system of translation-isometric coordinates x^1, x^2, \dots, x^{m+1} as in Theorem \hat{C} . A Ψ_a graph is a graph described by*

$$(x^1, x^2, \dots, x^m, u(x^1, x^2, \dots, x^m)).$$

For a Ψ_a graph, we take the defining function $\phi = u(x^1, x^2, \dots, x^m) - x^{m+1}$ which is compatible with $\{\Psi_a\}$. For $p \geq 0$, we define

$$(1.9) \quad H_p(u)(x^1, \dots, x^m) := H_{\phi,p}(x^1, \dots, x^m, u(x^1, x^2, \dots, x^m))$$

at (x^1, \dots, x^m) where $|d\phi|_* \neq 0$. Making use of translation-isometric coordinates, we can reformulate Theorem C as follows.

Theorem C'. *Suppose $(M, \langle \cdot, \cdot \rangle^*, dv_M)$ of dimension $m+1$ has isometric translations Ψ_a near $p_0 \in M$, transversal to a hypersurface Σ passing through p_0 . Take a system of translation-isometric coordinates x^1, x^2, \dots, x^{m+1} in a neighborhood V of p_0 such that $x^{m+1} = 0$ on Σ . Suppose u (v , resp.) : $\Sigma \cap V \rightarrow \mathbb{R}$ defines a graph $\{(x^1, x^2, \dots, x^m, u(x^1, x^2, \dots, x^m))\}$ ($\{(x^1, x^2, \dots, x^m, v(x^1, x^2, \dots, x^m))\}$, resp.) $\subset V$ such that $|d(u - x^{m+1})|_* \neq 0$ ($|d(v - x^{m+1})|_* \neq 0$, resp.). Assume*

- (1) $v \geq u$ on $\Sigma \cap V$ and $v(0, \dots, 0) = u(0, \dots, 0) = 0$;
- (2) For some $p \geq 0$, $H_p(v) \leq H_p(u)$ on $\Sigma \cap V$.

Moreover, assume the rank condition (1.7) holds near p_0 . Then $v \equiv u$ in a neighborhood of $p_0 \in \Sigma$.

Let M be the Heisenberg group H_n considered as a pseudohermitian manifold and hence a subriemannian manifold (see Appendix B for detailed explanation). Suppose two hypersurfaces Σ_1 and Σ_2 in Theorem C are (horizontal) graphs over the $x^1 x^2 \dots x^{2n}$ hyperplane, defined by v and u , resp.. We can take Ψ_a to be the translation in the last coordinate by the amount a : $\Psi_a(x^1, x^2, \dots, x^{2n-1}, x^{2n}, z) = (x^1, x^2, \dots, x^{2n-1}, x^{2n}, z + a)$. The defining functions ψ and ϕ for Σ_2 and Σ_1 , resp. are given by $v(x^1, x^2, \dots, x^{2n-1}, x^{2n}) - z$ and $u(x^1, x^2, \dots, x^{2n-1}, x^{2n}) - z$. We can then verify the assumption of Theorem C or Theorem C' ($m = 2n$) and that condition (2) for the case $p = 0$ is equivalent to $H_{\bar{F}}(v) \leq H_{\bar{F}}(u)$ (see (1.12) below) by identifying $H_{\bar{F}}(v)$ and $H_{\bar{F}}(u)$ with the mean curvature of ψ and ϕ , resp., with respect to a certain subriemannian manifold $(M, \langle \cdot, \cdot \rangle^*, dv_M)$ while condition (1) is the same. So Theorem C or Theorem C' includes the Heisenberg group case (see Theorem F below and its proof in Section 4 for more details).

Another situation is that two hypersurfaces Σ_1 and Σ_2 are tangent at p_0 vertically in H_n , i.e., the common tangent space at p_0 is a hyperplane E perpendicular to the $x^1 x^2 \dots x^{2n}$ hyperplane. We can then find a one-parameter family of Heisenberg translations in a direction of vector normal to this tangent space at p_0 . Let l_a denote the Heisenberg translation in the direction $\frac{\partial}{\partial x^{2n+1}}$:

$$(1.10) \quad l_a(x^1, x^2, \dots, x^{2n-1}, x^{2n}, z) := (x^1 + a, x^2, \dots, x^{2n-1}, x^{2n}, z - ax^{2n+1}).$$

Corollary D. *Suppose Σ_1 and Σ_2 are tangent at p_0 vertically in H_n with $n \geq 2$. Suppose the (Euclidean) unit normal to the tangent space at p_0 is $-\frac{\partial}{\partial x^{2n+1}}$ without loss of generality. For $p \geq 0$, we assume*

$$H_{\Sigma_2,p}(l_{a(q)}(q)) \leq H_{\Sigma_1,p}(q)$$

for $q \in \Sigma_1$ near p_0 and $l_{a(q)}(q) \in \Sigma_2$ with $a(q) \geq 0$. Then Σ_2 coincides with Σ_1 near p_0 .

As an example of Theorem C', we have the SMP for (horizontal) mean curvature of l_a graphs. See Corollary D' in Section 3. Observe that translation-isometric coordinates with respect to the Heisenberg translation in the direction $\frac{\partial}{\partial x^{2n+1}}$ are

closely related to coordinates for an intrinsic graph. Recall ([1]) that an intrinsic graph u in H_n is a hypersurface of the form $(0, \eta^2, \eta^3, \dots, \eta^{2n}, \tau) \circ (u(\eta^2, \eta^3, \dots, \eta^{2n}, \tau), 0, \dots, 0)$. Namely, it is parametrized by $\eta^2, \eta^3, \dots, \eta^{2n}, \tau$ so that

$$\begin{aligned} x^1 &= u(\eta^2, \eta^3, \dots, \eta^{2n}, \tau), \\ x^2 &= \eta^2, \dots, x^{2n} = \eta^{2n}, \\ z &= \tau + \eta^{n+1} u(\eta^2, \eta^3, \dots, \eta^{2n}, \tau) \end{aligned}$$

(see the Appendix for the definition of multiplication \circ in H_n). Let

$$\begin{aligned} \dot{e}_2 &:= \frac{\partial}{\partial \eta^2} + \eta^{n+2} \frac{\partial}{\partial \tau}, \dots, \dot{e}_n := \frac{\partial}{\partial \eta^n} + \eta^{2n} \frac{\partial}{\partial \tau}, \\ \dot{e}_{n+1}^u &:= \frac{\partial}{\partial \eta^{n+1}} - 2u \frac{\partial}{\partial \tau}, \\ \dot{e}_{n+2} &:= \frac{\partial}{\partial \eta^{n+2}} - \eta^2 \frac{\partial}{\partial \tau}, \dots, \dot{e}_{2n} := \frac{\partial}{\partial \eta^{2n}} - \eta^n \frac{\partial}{\partial \tau}. \end{aligned}$$

Define a vector-valued operator W^u by

$$W^u := (\dot{e}_2, \dots, \dot{e}_n, \dot{e}_{n+1}^u, \dot{e}_{n+2}, \dots, \dot{e}_{2n}).$$

The horizontal (or p -)mean curvature of an intrinsic graph u is given by

$$(1.11) \quad H_{u(\eta^2, \eta^3, \dots, \eta^{2n}, \tau) - x^1} = W^u \cdot \left(\frac{W^u(u)}{\sqrt{1 + |W^u(u)|^2}} \right)$$

at $(\eta^2, \eta^3, \dots, \eta^{2n}, \tau, u(\eta^2, \eta^3, \dots, \eta^{2n}, \tau))$ (see (3.21)). In Section 3 we provide more details and observe that an intrinsic graph is congruent with an l_a graph by a rotation in a certain situation. We therefore have the SMP for intrinsic graphs with constant horizontal (or p -)mean curvature as a special case of Theorem C'.

Theorem E. *Suppose $n \geq 2$. Let $v = v(\eta^2, \eta^3, \dots, \eta^{2n}, \tau)$, $u = u(\eta^2, \eta^3, \dots, \eta^{2n}, \tau)$ be two (C^∞ smooth) intrinsic graphs defined on a neighborhood U of $p_0 = (\eta_0^2, \eta_0^3, \dots, \eta_0^{2n}, \tau_0)$. Assume $v = u$ at p_0 where $\nu_{\eta^{n+1}} \neq 0$ and $u_{\eta^{n+1}} \neq 0$. Suppose $v \geq u$,*

$$W^v \cdot \left(\frac{W^v(v)}{\sqrt{1 + |W^v(v)|^2}} \right) \leq W^u \cdot \left(\frac{W^u(u)}{\sqrt{1 + |W^u(u)|^2}} \right)$$

in U , and either v or u has constant horizontal (or p -)mean curvature. Then $v \equiv u$ near p_0 .

We remark that the horizontal mean curvature operator of an intrinsic graph u does not belong to the type (1.1) since the second order coefficients contain u itself. So Theorem E does not follow directly from previous general theorems.

Let Ω be a (connected and open) domain of R^m . Let u, v be two C^2 smooth, real valued functions on Ω . Let \vec{F} be a C^1 smooth vector field on Ω . Define the Legendrian (or horizontal) normal $N_{\vec{F}}(u)$ ($N_{\vec{F}}(v)$, resp.) of u (v , resp.) by

$$N_{\vec{F}}(u) := \frac{\nabla u + \vec{F}}{|\nabla u + \vec{F}|}$$

($N_{\vec{F}}(v) := \frac{\nabla v + \vec{F}}{|\nabla v + \vec{F}|}$, resp.) at points where $\nabla u + \vec{F} \neq 0$ ($\nabla v + \vec{F} \neq 0$, resp.). Define the (generalized) horizontal (or p -) mean curvature $H_{\vec{F}}(u)$ ($H_{\vec{F}}(v)$, resp.) by

$$(1.12) \quad H_{\vec{F}}(u) := \operatorname{div} N_{\vec{F}}(u)$$

($H_{\vec{F}}(v) := \operatorname{div} N_{\vec{F}}(v)$, resp.). We call a point p_0 singular with respect to v if $\nabla v + \vec{F} = 0$ at p_0 . Denote the set of all singular points with respect to v by $S_{\vec{F}}(v)$.

In a neighborhood U of a nonsingular point $q_0 \in \Omega \setminus S_{\vec{F}}(v)$, let $N_1^\perp(v), N_2^\perp(v), \dots, N_{m-1}^\perp(v)$ be an orthonormal basis of the space perpendicular to $N_{\vec{F}}(v)$. Let $\mathcal{L}(N_1^\perp(v), N_2^\perp(v), \dots, N_{m-1}^\perp(v))$ denote the smallest C^∞ -module which contains $N_1^\perp(v), N_2^\perp(v), \dots, N_{m-1}^\perp(v)$, and is closed under the Lie bracket (see (7.4) in the Appendix for precise definition). The rank of $\mathcal{L}(N_1^\perp(v), N_2^\perp(v), \dots, N_{m-1}^\perp(v))$ at a point $q \in U$ is the dimension of the vector space spanned by the vectors $Z(q)$ for all $Z \in \mathcal{L}(N_1^\perp(v), N_2^\perp(v), \dots, N_{m-1}^\perp(v))$. The following result is a special, but important case of Theorem C or Theorem C' for degree of degeneracy $l = 1$.

Theorem F. *Suppose $m \geq 3$, $H_{\vec{F}}(v) \leq H_{\vec{F}}(u)$, $v \geq u$ in $U \subset R^m$, which is a nonsingular domain for both v and u , and $v = u$ at $p_0 \in U$. Assume in U an orthonormal basis $N_1^\perp(u), N_2^\perp(u), \dots, N_{m-1}^\perp(u)$ of the space perpendicular to $N_{\vec{F}}(u)$ exists and the rank of $\mathcal{L}(N_1^\perp(u), N_2^\perp(u), \dots, N_{m-1}^\perp(u))$ is constant m (similar condition for $N_{\vec{F}}(v)$, resp.). Then we have $v \equiv u$ in U .*

Write

$$(1.13) \quad N_\alpha^\perp(v) = \sum_{k=1}^m b_\alpha^k \partial_k.$$

Corollary G. *Suppose $m \geq 3$, $H_{\vec{F}}(u) \leq H_{\vec{F}}(v)$, $u \geq v$ in $U \subset R^m$, which is a nonsingular domain for both v and u , and $u = v$ at $p_0 \in U$. Assume there exists a pair of (α, β) , $\alpha \neq \beta$, such that*

$$(1.14) \quad \sum_{k < j} (\partial_k F_j - \partial_j F_k) (b_\alpha^k b_\beta^j - b_\alpha^j b_\beta^k) \neq 0$$

(similar condition for $N_\alpha^\perp(u)$, resp.) in U . Then we have $u \equiv v$ in U .

Corollary H. *Suppose $m \geq 3$, $H_{\vec{F}}(u) \leq H_{\vec{F}}(v)$, $u \geq v$ in $U \subset R^m$, which is a nonsingular domain for both u and v , and $u = v$ at $p_0 \in U$. Assume*

$$(1.15) \quad \operatorname{rank}(\partial_k F_j - \partial_j F_k) \geq 3$$

in U . Then we have $u \equiv v$ in U .

Corollary I. *Suppose $m = 2n \geq 4$, $H_{\vec{F}}(u) \leq H_{\vec{F}}(v)$, $u \geq v$ in $U \subset R^{2n}$, which is a nonsingular domain for both v and u , and $u = v$ at $p_0 \in U$. Assume $\vec{F} = (-x^2, x^1, \dots, -x^{2n}, x^{2n-1})$. Then we have $u \equiv v$ in U .*

Corollary I provides the SMP of so called horizontal (or p -) mean curvature for hypersurfaces given by graphs over a domain of the $x^1 x^2 \dots x^{2n}$ hyperplane in the Heisenberg group H_n (identified with R^{2n+1} as a set).

We remark that when $m = 2$, Corollary I does not hold. That is, the SMP of horizontal (p -)mean curvature for surfaces in H_1 does not hold (although the maximum principle holds; see [12] or Theorem C'' below) as shown by the following example. Let $u = x^1 x^2 + (x^2)^2$ and $v = x^1 x^2$. It follows from (1.12) that $H_{\vec{F}}(u) = H_{\vec{F}}(v) = 0$ in a nonsingular domain for $\vec{F} = (-x^2, x^1)$. Observe that $u \geq v$ in $U = \{x^1 > 0, x^1 + x^2 > 0\}$, a nonsingular domain for both u and v , and $u = v$ at $(x^1, 0) \in U$. But apparently $u \neq v$ in $U \setminus \{(x^1, 0) : x^1 > 0\}$.

Define \vec{G}^b for $\vec{G} = (G_1, \dots, G_m)$ by

$$\vec{G}^b := \left(\sum_{k=1}^m a^{1k} G_k, \sum_{k=1}^m a^{2k} G_k, \dots, \sum_{k=1}^m a^{mk} G_k \right)$$

where a^{jk} 's are real constants such that $a^{jk} + a^{kj} = 0$ for $1 \leq j, k \leq m$. Note that $\vec{G}^b = \vec{G}^*$ for $m = 2n$, $a^{2j-1, 2j} = -a^{2j, 2j-1} = 1$, $1 \leq j \leq n$, $a^{jk} = 0$ otherwise. When p_0 is an isolated singular point of v , we still have the SMP.

Theorem J. *Suppose $m \geq 3$, $v \geq u$ in $\Omega \subset R^m$, such that $\Omega \cap S_{\vec{F}}(u) = \{p_0\}$ ($\Omega \cap S_{\vec{F}}(v) = \{p_0\}$, resp.) and $v = u$ at p_0 . Suppose $\mathcal{H}_{m-1}(\overline{S_{\vec{F}}(v)}) = 0$ ($\mathcal{H}_{m-1}(\overline{S_{\vec{F}}(u)}) = 0$, resp.) and $\operatorname{div} \vec{F}^b > 0$ (or $\operatorname{div} \vec{F}^b < 0$). Assume $H_{\vec{F}}(v) \leq H_{\vec{F}}(u)$ in $\Omega \setminus \{\{p_0\} \cup S_{\vec{F}}(v)\}$ ($\Omega \setminus \{\{p_0\} \cup S_{\vec{F}}(u)\}$, resp.) and for each point $p \in \Omega \setminus \{p_0\}$, there is a neighborhood U of p in which an orthonormal basis $N_1^\perp(u)$, $N_2^\perp(u)$, ..., $N_{m-1}^\perp(u)$ of the space perpendicular to $N_{\vec{F}}(u)$ exists and the rank of $\mathcal{L}(N_1^\perp(u), N_2^\perp(u), \dots, N_{m-1}^\perp(u))$ is constant m (similar condition for $N_{\vec{F}}(v)$, resp.). Then we have $v \equiv u$ in Ω .*

In the proof of Theorem J, we need to apply the following version of the usual maximum principle (in the case of removable singularity) for $H_{\vec{F}}$.

Theorem C'' (an extension of Theorem C' in [12]). *For a bounded domain Ω in R^m with $m \geq 2$, let $v, u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ satisfy*

$$\begin{aligned} H_{\vec{F}}(v) &\leq H_{\vec{F}}(u) \text{ in } \Omega \setminus \{S_{\vec{F}}(u) \cup S_{\vec{F}}(v)\} \\ v &\geq u \text{ on } \partial\Omega \end{aligned}$$

Suppose $\mathcal{H}_{m-1}(\overline{S_{\vec{F}}(u) \cup S_{\vec{F}}(v)}) = 0$ and $\vec{F} \in C^1(\Omega) \cap C^0(\bar{\Omega})$ satisfies $\operatorname{div} \vec{F}^b > 0$ ($\operatorname{div} \vec{F}^b < 0$, resp.) in Ω . Then $v \geq u$ in Ω .

We remark that the condition $\operatorname{div} \vec{F}^b > 0$ or < 0 was first used to extend uniqueness results from even dimension to arbitrary dimension in [11]. In view of Corollary G (Corollary I, resp.), the condition on $N_{\vec{F}}(v)$ or $N_{\vec{F}}(u)$ in Theorem F' and Theorem J can be replaced by (1.14) ($\vec{F} = (-x^2, x^1, \dots, -x^{2n}, x^{2n-1})$, resp. for $m = 2n$). Let H_n denote the Heisenberg group of dimension $2n + 1$. As a set, H_n is $C^n \times R$ or $R^{2n} \times R$. For a hypersurface Σ in H_n (which may not be a graph over R^{2n}), the horizontal (or p -) mean curvature H_Σ of Σ for a defining function ψ is given by

$$(1.16) \quad H_\Sigma := \operatorname{div}_b \frac{\nabla_b \psi}{|\nabla_b \psi|}$$

where ∇_b and div_b denote subgradient and subdivergence in H_n , resp.. See Subsection B of the Appendix for equivalent definitions of mean curvature in subriemannian geometry. Note that for a graph Σ over R^{2n} defined by u , H_Σ may be different from $H_{\vec{F}}(u)$ ($\vec{F} = (-x^2, x^1, \dots, -x^{2n}, x^{2n-1})$) by sign. In fact, if we replace ψ by $-\psi$ in (1.16), H_Σ becomes $-H_\Sigma$.

For the boundary Σ of a (C^2 smooth, say) bounded domain Ω in H_n , we choose a defining function ψ for Σ , such that $\psi < 0$ in Ω . In this way H_Σ is a positive constant for a Pansu sphere given by the union of all the geodesics of positive constant curvature joining the two poles (see, e.g., [35] for the $m = 2$ case and [33] for the higher dimensional case).

Theorem K. *Let Σ_1 and Σ_2 be two C^2 smooth, connected, orientable, closed hypersurfaces of constant horizontal (p -) mean curvature H_{Σ_1} and H_{Σ_2} , resp. in H_n , $n \geq 2$. Suppose Σ_2 is inscribed in Σ_1 , i.e., Σ_2 is contained in the closure of the inside of Σ_1 and $\Sigma_1 \cap \Sigma_2$ is not empty. Assume $H_{\Sigma_2} \leq H_{\Sigma_1}$ and $\Sigma_1 \cap \Sigma_2$ contains a nonsingular (with respect to both Σ_1 and Σ_2) point or an isolated singular point of Σ_1 (Σ_2 , resp.). Moreover, assume either Σ_1 or Σ_2 has only isolated singular points. Then $\Sigma_1 \equiv \Sigma_2$.*

For further applications we need to extend Theorem J to hypersurfaces of a subriemannian manifold having isometric translations, with an isolated singular point in touch. A point $\tilde{q} \in \tilde{\Sigma}$, a hypersurface of a subriemannian manifold, is called singular if $\xi \subset T\tilde{\Sigma}$ at \tilde{q} (this is equivalent to $|d\phi|_* = 0$ at \tilde{q} for a defining function ϕ of $\tilde{\Sigma}$ mentioned previously). For $\tilde{\Sigma}$ being a graph described by $(x^1, x^2, \dots, x^m, w(x^1, x^2, \dots, x^m))$, $(x^1, x^2, \dots, x^m) \in D$, in local coordinates, we call a point q in D singular for w if $(q, w(q))$ is a singular point of $\tilde{\Sigma}$. Denote the set of all singular points in D for w by $S_D(w)$ or $S(w)$ if the domain of w is clear in the context.

Theorem J'. *Suppose $(M, \langle \cdot, \cdot \rangle^*, dv_M)$ of dimension $m+1$ has isometric translations Ψ_a near $p_0 \in M$, transversal to a hypersurface Σ passing through p_0 . Take a system of translation-isometric coordinates x^1, x^2, \dots, x^{m+1} in a neighborhood Ω of p_0 such that $x^{m+1} = 0$ on (connected) $\Sigma \cap \Omega$. Suppose u (v , resp.) : $\Sigma \cap \Omega \rightarrow \mathbb{R}$ defines a graph $\{(x^1, x^2, \dots, x^m, u(x^1, x^2, \dots, x^m))\}$ ($\{(x^1, x^2, \dots, x^m, v(x^1, x^2, \dots, x^m))\}$, resp.) $\subset \Omega$. Assume*

(1) $v \geq u$ on $\Sigma \cap \Omega$ such that $S_{\Sigma \cap \Omega}(u) = \{p_0\}$ ($S_{\Sigma \cap \Omega}(v) = \{p_0\}$, resp.), $p_0 = (0, \dots, 0)$, and $v(p_0) = u(p_0) = 0$;

(2) $H(v) \leq H(u)$ in $(\Sigma \cap \Omega) \setminus \{\{p_0\} \cup S_{\Sigma \cap \Omega}(v)\}$ ($(\Sigma \cap \Omega) \setminus \{\{p_0\} \cup S_{\Sigma \cap \Omega}(u)\}$, resp.).

Suppose $\mathcal{H}_{m-1}(\overline{S_{\Sigma \cap \Omega}(v)}) = 0$ ($\mathcal{H}_{m-1}(\overline{S_{\Sigma \cap \Omega}(u)}) = 0$, resp.). Moreover, we assume the rank condition (1.7) holds. Then we have $v \equiv u$ in $\Sigma \cap \Omega$.

Theorem C''. *Suppose $(M, \langle \cdot, \cdot \rangle^*, dv_M)$ of dimension $m+1$ has isometric translations Ψ_a near $p_0 \in M$, transversal to a hypersurface Σ passing through p_0 . Take a system of translation-isometric coordinates x^1, x^2, \dots, x^{m+1} in an open neighborhood Ω of p_0 such that $x^{m+1} = 0$ on $\Sigma \cap \Omega$. Let $V \subset \bar{V} \subset \Omega$ be a smaller open neighborhood of p_0 . Let $v, u \in C^2(\Sigma \cap V) \cap C^0(\overline{\Sigma \cap V})$ define graphs in Ω and satisfy*

$$H(v) \leq H(u) \text{ in } (\Sigma \cap V) \setminus \{S_{\Sigma \cap V}(u) \cup S_{\Sigma \cap V}(v)\},$$

$$v \geq u \text{ on } \partial(\Sigma \cap V).$$

Assume the rank condition (1.7) and $\mathcal{H}_{m-1}(\overline{S_{\Sigma \cap V}(u) \cup S_{\Sigma \cap V}(v)}) = 0$. Then $v \geq u$ in $\Sigma \cap V$.

We can now apply Theorem C (or C') and Theorem J' to prove a rigidity result for minimal hypersurfaces in a Heisenberg cylinder $(H_n \setminus \{0\}, \rho^{-2}\Theta)$ with $n \geq 2$. Here Θ denotes the standard Heisenberg contact form:

$$\Theta := dz + \sum_{j=1}^n (x^j dx^{n+j} - x^{n+j} dx^j).$$

The associated Heisenberg distance function ρ reads

$$\rho := [(\sum_{K=1}^{2n} (x^K)^2)^2 + 4z^2]^{1/4}.$$

In Section 6 we discuss some basic geometry associated to the contact form $\rho^{-2}\Theta$ (using x_j, y_j instead of x^j, x^{n+j} and both interchangeably) before proving the following result.

Theorem L. *Let Σ be a closed (compact with no boundary) immersed hypersurface in a Heisenberg cylinder $(H_n \setminus \{0\}, \rho^{-2}\Theta)$ with $n \geq 2$. Suppose either*

(a) $H_\Sigma \leq 0$ or

(b) $H_\Sigma \geq 0$ and the interior region of Σ contains the origin of H_n

holds. Then Σ must be a Heisenberg sphere defined by $\rho^4 = c$ for some constant $c > 0$. In particular, $H_\Sigma \equiv 0$.

Corollary M. *There does not exist a closed immersed hypersurface of positive constant horizontal (p -)mean curvature in a Heisenberg cylinder $(H_n \setminus \{0\}, \rho^{-2}\Theta)$ with $n \geq 2$, whose interior region contains the origin.*

Let φ be a continuous function of $\tau \in [0, \infty)$. We have the following nonexistence result (pseudo-halfspace theorem).

Theorem N. *Let Ω be a domain of $H_n, n \geq 2$, defined by either $z > \varphi(\sqrt{x_1^2 + \dots + x_{2n}^2})$ or $x_1 > \varphi(\sqrt{x_2^2 + \dots + x_{2n}^2 + z^2})$ where $\lim_{\tau \rightarrow \infty} \varphi(\tau) = \infty$. Then there does not exist any horizontal (p -) minimal hypersurface properly immersed in Ω .*

The simplest example for Theorem N is $\varphi(\tau) = a\tau$ with $a > 0$. Call associated domains wedge-shaped. Theorem N tells us nonexistence of horizontal (p -) minimal hypersurfaces in wedge-shaped domains. But Theorem N does not hold for the case $a = 0$. That is, halfspace theorem does not hold since there are catenoid type horizontal (p -)minimal hypersurfaces with finite height ([34]) in H_n for $n \geq 2$. On the other hand, we do have halfspace theorem for H_1 (see [9]). Hoffman and Meeks ([22]) first proved such a halfspace theorem for R^3 . It fails for R^n with $n \geq 4$. But above type of pseudo-halfspace theorem still holds for R^n with $n \geq 4$ by a similar reasoning.

There is another notion of mean curvature, called Levi-mean curvature, in the study of real hypersurfaces in C^n . We would like to remark that the SMP for such mean curvature operators (generalized to pseudoconvex fully nonlinear Levi-type curvature operators) has been proved by Montanari and Lanconelli ([27]).

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2. PROOFS OF THEOREMS A, A', \tilde{A} , AND B

Proof. (of Theorem A) From the definition of Q (see 1.1), we compute the difference of $Q\phi$ and $Q\psi$ as follows:

$$(2.1) \quad \begin{aligned} Q\phi - Q\psi &= a^{ij}(x, D\phi)D_{ij}(\phi - \psi) + (a^{ij}(x, D\phi) - a^{ij}(x, D\psi))D_{ij}\psi \\ &\quad + (b(x, \phi, D\phi) - b(x, \phi, D\psi)) + (b(x, \phi, D\psi) - b(x, \psi, D\psi)) \geq 0 \end{aligned}$$

by assumption. Writing

$$\begin{aligned} w &= \phi - \psi, \\ \tilde{a}^{ij}(x) &= a^{ij}(x, D\phi), \\ (a^{ij}(x, D\phi) - a^{ij}(x, D\psi))D_{ij}\psi + (b(x, \phi, D\phi) - b(x, \phi, D\psi)) &= \tilde{b}^i(x)D_iw, \\ b(x, \phi, D\psi) - b(x, \psi, D\psi) &= \tilde{a}(x)w, \end{aligned}$$

we get

$$Lw := \tilde{a}^{ij}(x)D_{ij}w + \tilde{b}^i(x)D_iw + \tilde{a}(x)w \geq 0.$$

Noting that the quadratic form $(\tilde{a}^{ij}(x))$ is nonnegative by condition (1) and $\tilde{a}(x) \leq 0$ by condition (2), we can then apply Theorem A7 in the Appendix (Theorem 3.1 in [5]) to complete the proof. \square

Proof. (of Theorem A') Write $e_i = \alpha_i^l \partial_l$. Compute

$$a^{ij}e_i e_j \phi = a^{ij} \alpha_i^l \alpha_j^k \partial_l \partial_k \phi + a^{ij} \alpha_i^l (\partial_l \alpha_j^k) \partial_k \phi.$$

Observe that $\tilde{a}^{lk} := a^{ij} \alpha_i^l \alpha_j^k$ satisfies $\tilde{a}^{lk} = \tilde{a}^{kl}$ since $a^{ij} = a^{ji}$, and (\tilde{a}^{lk}) is nonnegative since (a^{ij}) is nonnegative. Note also that coefficients of first derivatives $\partial_k \phi$ do not rely on the variable ϕ . So $Q'\phi$ is of the form (1.1) for a certain $Q\phi$ which satisfies the conditions (1), (2), (3) in Theorem A. Thus the conclusion follows from Theorem A. \square

Proof. (of Theorem \tilde{A}) It suffices to show (1.3) implies condition (3). \square

Given a subriemannian manifold $(M, \langle \cdot, \cdot \rangle^*)$ of dimension $m+1$, we recall that $G : T^*M \rightarrow TM$ is defined by $\omega(G(\eta)) = \langle \omega, \eta \rangle^*$ for $\omega, \eta \in T^*M$. Define

$$\langle v, w \rangle_* := \langle \eta, \zeta \rangle^*$$

for $v, w \in \xi := \text{Range}(G)$ and any choice $\eta \in G^{-1}(v)$, $\zeta \in G^{-1}(w)$. It is easy to see that $\langle \cdot, \cdot \rangle_*$ is well defined. Assume

$$\dim \xi = \text{constant } m+1-l$$

for an integer l , $0 \leq l \leq m+1$. So $\langle \cdot, \cdot \rangle_*$ is positive definite (for $l \leq m$) on the vector bundle ξ . There exist (C^∞) smooth local sections v_1, \dots, v_{m+1-l} of ξ ,

orthonormal with respect to $\langle \cdot, \cdot \rangle_*$ (for \cdot). We choose any smooth element $\eta^j \in G^{-1}(v_j)$. It follows that

$$\langle \eta^i, \eta^j \rangle_* = \delta_{ij}$$

for $1 \leq i, j \leq m+1-l$. Now given a background volume form dv_M , for $l \geq 1$, we then choose smooth independent sections $\eta^{m+2-l}, \dots, \eta^{m+1}$ of $\text{Ker} G \subset T^*M$ such that $\langle \eta^j, \cdot \rangle_* = \langle \cdot, \eta^j \rangle_* = 0$ for $m+2-l \leq j \leq m+1$ and

$$dv_M = \eta^1 \wedge \dots \wedge \eta^{m+1}$$

(note that we have freedom to choose a scalar multiple of $\eta^{m+2-l}, \dots, \eta^{m+1}$). For $l = 0$, $\langle \cdot, \cdot \rangle_*$ is a Riemannian metric on T^*M and $\eta^1 \wedge \dots \wedge \eta^{m+1}$ is the Riemannian volume form (up to a sign). So a given volume form dv_M is a nonzero scalar multiple of $\eta^1 \wedge \dots \wedge \eta^{m+1}$. We have shown

Lemma 2.1. *Let $(M, \langle \cdot, \cdot \rangle_*, dv_M)$ be a subriemannian manifold of dimension $m+1$ with a background volume form dv_M . Assume*

$$\dim \xi = \text{constant } m+1-l.$$

(cf. (1.6)) *Then locally we can choose a suitable (C^∞ smooth) coframe $\eta^1, \dots, \eta^{m+1}$ such that $\langle \eta^i, \eta^j \rangle_* = \delta_{ij}$ for $1 \leq i, j \leq m+1-l$, $\langle \eta^i, \eta^j \rangle_* = 0$ otherwise, and $dv_M = \eta^1 \wedge \dots \wedge \eta^{m+1}$ for $1 \leq l \leq m+1$ while dv_M is a nonzero scalar multiple of Riemannian volume form for $l = 0$.*

Proof. (of Theorem B) Take a coframe $\omega^1, \omega^2, \dots, \omega^{m+1}$ in T^*M (near p_0), such that $dv_M = \omega^1 \wedge \omega^2 \wedge \dots \wedge \omega^{m+1}$. Let e_1, e_2, \dots, e_{m+1} be the dual frame in TM . Compute

$$\begin{aligned} (2.2) \quad d\phi \lrcorner dv_M &= (e_i \phi) \omega^i \lrcorner \omega^1 \wedge \omega^2 \wedge \dots \wedge \omega^{m+1} \\ &= (e_i \phi) g^{ij} (-1)^{j-1} \omega^1 \wedge \dots \wedge \hat{\omega}^j \wedge \dots \wedge \omega^{m+1} \end{aligned}$$

where $g^{ij} = \langle \omega^i, \omega^j \rangle_*$ and $\hat{\omega}^j$ means deleting ω^j . From (2.2) and $|d\phi|_* \neq 0$ at p_0 , we then compute (near p_0)

$$\begin{aligned} (2.3) \quad & d\left(\frac{d\phi}{|d\phi|_*^{1-p}} \lrcorner dv_M\right) \\ &= d\left(\frac{e_i \phi}{|d\phi|_*^{1-p}}\right) g^{ij} (-1)^{j-1} \omega^1 \wedge \dots \wedge \hat{\omega}^j \wedge \dots \wedge \omega^{m+1} \\ &\quad + \frac{e_i \phi}{|d\phi|_*^{1-p}} (-1)^{j-1} d(g^{ij} \omega^1 \wedge \dots \wedge \hat{\omega}^j \wedge \dots \wedge \omega^{m+1}). \end{aligned}$$

The first term of the right-hand side in (2.3) is the term of second order in ϕ . We compute it as follows:

$$\begin{aligned} (2.4) \quad & d\left(\frac{e_i \phi}{|d\phi|_*^{1-p}}\right) g^{ij} (-1)^{j-1} \omega^1 \wedge \dots \wedge \hat{\omega}^j \wedge \dots \wedge \omega^{m+1} \\ &= e_k \left(\frac{e_i \phi}{|d\phi|_*^{1-p}}\right) \omega^k g^{ij} (-1)^{j-1} \omega^1 \wedge \dots \wedge \hat{\omega}^j \wedge \dots \wedge \omega^{m+1} \\ &= e_j \left(\frac{e_i \phi}{|d\phi|_*^{1-p}}\right) g^{ij} dv_M. \end{aligned}$$

So by (1.4), (2.3), and (2.4), we have

$$(2.5) \quad H_{\phi,p} = e_j \left(\frac{e_i \phi}{|d\phi|_*^{1-p}} \right) g^{ij} + \text{first order terms in } \phi.$$

Note that the first order terms in (2.5) do not depend on the variable ϕ itself.
Write

$$(2.6) \quad \begin{aligned} e_j \left(\frac{e_i \phi}{|d\phi|_*^{1-p}} \right) g^{ij} &= g^{ij} \frac{e_i e_j \phi}{|d\phi|_*^{1-p}} \\ &+ (p-1) \frac{g^{ij} (e_i \phi) (e_k \phi) (e_j e_l \phi) g^{kl}}{|d\phi|_*^{3-p}} \\ &+ (p-1) \frac{g^{ij} (e_i \phi) (e_k \phi) (e_l \phi) (e_j g^{kl})}{2|d\phi|_*^{3-p}}. \end{aligned}$$

Define the second order operator $a^{ij}(x, e_1 \phi, \dots, e_{m+1} \phi) e_i e_j$ by

$$(2.7) \quad a^{ij}(x, e_1 \phi, \dots, e_{m+1} \phi) := \frac{g^{ij}(x)}{|d\phi|_*^{1-p}} + (p-1) \frac{(e^i \phi)(e^j \phi)}{|d\phi|_*^{3-p}}$$

where $e^i := g^{ij} e_j$. Observe that $H_{\phi,p}$ is independent of the choice of (co)frames. By Lemma 2.1, we can choose a suitable coframe (field), denoted as $\tilde{\omega}^1, \tilde{\omega}^2, \dots, \tilde{\omega}^{m+1}$, such that $\tilde{g}^{ij} = \delta_{ij}$ for $1 \leq i, j \leq m+1-l$, $\tilde{g}^{ij} = 0$ otherwise, and $dv_M = \tilde{\omega}^1 \wedge \tilde{\omega}^2 \wedge \dots \wedge \tilde{\omega}^{m+1}$ for $l \geq 1$. If $l = 0$, dv_M is a nonzero scalar multiple of the Riemannian volume form $\tilde{\omega}^1 \wedge \tilde{\omega}^2 \wedge \dots \wedge \tilde{\omega}^{m+1}$. So we have the same form of second order term and the later argument still works. Let $\{\tilde{e}_1, \dots, \tilde{e}_{m+1}\}$ be dual to $\{\tilde{\omega}^1, \tilde{\omega}^2, \dots, \tilde{\omega}^{m+1}\}$. Let

$$(2.8) \quad \tilde{e}_{m+1-l} := \frac{1}{|d\phi(x)|_*} \sum_{j=1}^{m+1-l} \tilde{e}_j \phi(x) \tilde{e}_j \in \xi$$

where $\xi := \text{Range}(G)$ is spanned by the orthonormal basis $\tilde{e}_1, \dots, \tilde{e}_{m+1-l}$. Choose another system of orthonormal vectors $\tilde{e}_1, \dots, \tilde{e}_{m-l}$ perpendicular to \tilde{e}_{m+1-l} in ξ . Also let $\tilde{e}_{m+2-l} = \tilde{e}_{m+2-l}, \dots, \tilde{e}_{m+1} = \tilde{e}_{m+1}$. Consider

$$\begin{aligned} \tilde{a}^{ij}(x) &: = a^{ij}(x, \tilde{e}_1 \phi(x), \dots, \tilde{e}_{m+1} \phi(x)) \\ &= \frac{\delta_{ij}}{|d\phi(x)|_*^{1-p}} + (p-1) \frac{\tilde{e}_i \phi(x) \tilde{e}_j \phi(x)}{|d\phi(x)|_*^{3-p}} \end{aligned}$$

(viewed as a function of x) for $1 \leq i, j \leq m+1-l$; $= (p-1) \frac{(\tilde{e}_i \phi)(\tilde{e}_j \phi)}{|d\phi|_*^{3-p}}$ otherwise. Compute

$$\begin{aligned}
 (2.9) \quad & \tilde{a}^{ij}(x) \tilde{e}_i \tilde{e}_j \\
 &= \frac{1}{|d\phi(x)|_*^{1-p}} \left\{ \sum_{j=1}^{m+1-l} \tilde{e}_j^2 + (p-1) \sum_{i,j=1}^{m+1} \frac{\tilde{e}_i \phi(x) \tilde{e}_j \phi(x)}{|d\phi(x)|_*^2} \tilde{e}_i \tilde{e}_j \right\} \\
 &= \frac{1}{|d\phi(x)|_*^{1-p}} \left\{ \sum_{j=1}^{m+1-l} \check{e}_j^2 + \text{first order} + (p-1) \check{e}_{m+1-l}^2 + \text{first order} \right\} \\
 &= \frac{1}{|d\phi(x)|_*^{1-p}} \left\{ \sum_{j=1}^{m-l} \check{e}_j^2 + p \check{e}_{m+1-l}^2 \right\} + \text{first order} \\
 &= \sum_{j=1}^{m-l} X_j^2 + \left(\frac{\sqrt{p}}{|d\phi(x)|_*^{(1-p)/2}} \check{e}_{m+1-l} \right)^2 + \text{first order}
 \end{aligned}$$

where

$$X_j := \frac{\check{e}_j}{|d\phi(x)|_*^{(1-p)/2}}.$$

Observe that $\check{e}_j \phi = 0$, and hence we have

$$(2.10) \quad X_j \in \xi \cap T\{\phi = 0\}.$$

i.e., X_j lies in the tangent space of hypersurface defined by $\phi = 0$. It is not hard to see that (a^{ij}) is symmetric and nonnegative by Cauchy-Schwarz inequality. In view of (2.4), (2.5), and (2.6), we learn that $H_{\phi,p} + b(x, \phi, D\phi)$ is an operator of type $Q'\phi$ in (1.2). Observe that condition (3) in Theorem A' holds by (2.9). By assumption we have

$$(2.11) \quad \text{rank}(\mathcal{L}(X_1, \dots, X_{m-l})) = m.$$

Since $\dim\{\phi = 0\} = m$, the integral curves of all $Z \in \mathcal{L}(X_1, \dots, X_{m-l})$ will cover a neighborhood of $\{\phi = 0\}$ by (2.10) and (2.11). (a) follows from Theorem A'.

In case $p > 0$, $X_1, X_2, \dots, X_{m-l}, \frac{\sqrt{p}}{|d\phi(x)|_*^{(1-p)/2}} \check{e}_{m+1-l}$ form a basis of ξ . By the assumption $\text{rank}(\mathcal{L}(\xi)) = m+1$, the integral curves of all $Z \in \mathcal{L}(\xi)$ will cover a neighborhood of p_0 in M . (b) follows from (2.9) and Theorem A'. \square

3. GRAPHS UNDER SYMMETRY AND PROOFS OF THEOREMS C, \hat{C} , C' , E AND COROLLARY D

Next we want to study when the conditions in Theorem B are satisfied. Let us start with a general subriemannian manifold $(M, \langle \cdot, \cdot \rangle^*)$ where $\langle \cdot, \cdot \rangle^*$ is a nonnegative definite inner product on T^*M . Take a background volume form dv_M . Let Σ_0 be a (local) hypersurface in M . Consider a one-parameter family of diffeomorphisms $\Psi_a : M \rightarrow M$, i.e., $\Psi_0 = \text{Identity}$, $\Psi_{a+b} = \Psi_a \circ \Psi_b$. We ask when the hypersurface $\Sigma_a := \Psi_a(\Sigma_0)$ has the same mean curvature (function) as Σ_0 .

Proposition 3.1. *Suppose $(M, < \cdot, \cdot >^*, dv_M)$ has isometric translations Ψ_a near $p_0 \in M$, compatible with a defining function ϕ (see Section 1 for the definition). Then for any $p \geq 0$, we have $H_{\phi,p}(\Psi_a(x)) = H_{\phi,p}(x)$.*

Proof. Recall that $H_{\phi,p}$ is defined by

$$d\left(\frac{d\phi}{|d\phi|_*^{1-p}} \lrcorner dv_M\right) := H_{\phi,p} dv_M.$$

Pulling back the above identity by Ψ_a , we get

$$(3.1) \quad d\left(\frac{d(\phi \circ \Psi_a)}{|d(\phi \circ \Psi_a)|_*^{1-p}} \lrcorner dv_M\right) = (H_{\phi,p} \circ \Psi_a) dv_M$$

by (a) and (b) in Definition 1.1. By the compatibility of Ψ_a with ϕ (see (1.8)), we compute the left hand side of (3.1):

$$(3.2) \quad \begin{aligned} & d\left(\frac{d(\phi \circ \Psi_a)}{|d(\phi \circ \Psi_a)|_*^{1-p}} \lrcorner dv_M\right) \\ &= d\left(\frac{d\phi}{|d\phi|_*^{1-p}} \lrcorner dv_M\right) = H_{\phi,p} dv_M. \end{aligned}$$

Comparing (3.1) with (3.2), we obtain $H_{\phi,p} \circ \Psi_a = H_{\phi,p}$. □

Proof. (of Theorem C) For any x near p_0 there exist a number b and $q \in \Sigma_1(\cap U)$ such that

$$x = \Psi_b(q).$$

Let $\tilde{q} := \Psi_{a(q)}(q) \in \Sigma_2$ with $a(q) \geq 0$ by condition (1). Choose defining functions ψ and ϕ compatible with Ψ_a : first define $\psi(\Psi_a(x)) = \psi(x) - a$ ($\phi(\Psi_a(x)) = \phi(x) - a$, resp.) for $x \in \Sigma_2$ (Σ_1 , resp.). The same formula then holds for any x near p_0 . We compute

$$(3.3) \quad \begin{aligned} \psi(x) &= \psi(\Psi_b(q)) = \psi(\Psi_{b-a(q)}(\tilde{q})) \\ &= \psi(\tilde{q}) - (b - a(q)) \\ &= 0 - b + a(q) \quad (\text{since } \tilde{q} \in \Sigma_2) \\ &\geq \phi(q) - b \quad (\text{since } q \in \Sigma_1 \text{ and } a(q) \geq 0) \\ &= \phi(\Psi_{-b}(x)) - b \\ &= \phi(x) - (-b) - b = \phi(x). \end{aligned}$$

On the other hand, we compute

$$(3.4) \quad \begin{aligned} H_{\psi,p}(x) &= H_{\psi,p}(\Psi_b(q)) = H_{\psi,p}(\Psi_{b-a(q)}(\tilde{q})) \\ &= H_{\psi,p}(\tilde{q}) \quad (\text{by Proposition 3.1}) \\ &= H_{\Sigma_2,p}(\tilde{q}) \quad (\text{by Proposition B.1 in the Appendix}) \\ &= H_{\Sigma_2,p}(\Psi_{a(q)}(q)) \\ &\leq H_{\Sigma_1,p}(q) \quad (\text{by condition (2)}) \\ &= H_{\phi,p}(q) \quad (\text{by Proposition B.1}) \\ &= H_{\phi,p}(\Psi_b(q)) \quad (\text{by Proposition 3.1}) \\ &= H_{\phi,p}(x). \end{aligned}$$

The result follows from (3.3), (3.4), and Theorem B with $b \equiv 0$. \square

Proof. (of Corollary D) Observe that the Heisenberg group H_n can be viewed as a subriemannian manifold. See Subsection B in the Appendix. $\{l_a\}$, the Heisenberg translations, is a one-parameter family of symmetries transversal to Σ_1 and Σ_2 near p_0 . Also we can choose ϕ and ψ to be defining functions for hypersurfaces Σ_1 and Σ_2 , resp. (I.e., Σ_1 (Σ_2 , resp.) is defined by $\phi = 0$ ($\psi = 0$, resp.)), compatible with $\{l_a\}$.

We claim that $\text{rank}(\mathcal{L}(\xi \cap T\Sigma_1)) = m$ near p_0 . Observe that $\dim \xi \cap T\Sigma_1 = 2n - 1 \geq 3$ for $n \geq 2$. We can find a J -invariant pair of nonzero vectors X, JX in $\xi \cap T\Sigma_1$. The Lie bracket $[X, JX]$ generates the direction out of contact distribution ξ . Therefore $\text{rank}(\mathcal{L}(\xi \cap T\Sigma_1)) = 2n = m$ near p_0 . We then conclude the result by Theorem C. \square

Proof. (of Theorem C) Take any system of local coordinates $\hat{x}^1, \hat{x}^2, \dots, \hat{x}^m$ on $\Sigma \cap U$ where U is a neighborhood of p_0 , such that $\hat{x}^j(p_0) = 0$. Observe that $V := \cup_{a \in (-\delta', \delta')} \Psi_a(\Sigma \cap U)$ for smaller $\delta' < \delta$ is a neighborhood of p_0 due to transversality of Ψ_a to Σ . Define the last coordinate x^{m+1} and x^1, x^2, \dots, x^m in V by

$$(3.5) \quad \begin{aligned} (1) \quad x^{m+1}(\Psi_a(\hat{q})) &= a, \\ (2) \quad x^j(\Psi_a(\hat{q})) &= \hat{x}^j(\hat{q}), \quad 1 \leq j \leq m \end{aligned}$$

for any $a \in (-\delta', \delta')$ and any $\hat{q} \in \Sigma \cap U$. Write an arbitrary point $q \in V$ as $q = \Psi_b(\hat{q})$ for $b \in (-\delta', \delta')$. Choose a such that $a + b \in (-\delta', \delta')$. Compute

$$(3.6) \quad \begin{aligned} x^j(\Psi_a(q)) &= x^j(\Psi_a(\Psi_b(\hat{q}))) \\ &= x^j(\Psi_{a+b}(\hat{q})) \\ &= \hat{x}^j(\hat{q}) \quad (\text{by (2) of (3.5)}) \\ &= x^j(\Psi_b(\hat{q})) \quad (\text{by (2) of (3.5)}) \\ &= x^j(q). \end{aligned}$$

Similarly we have

$$(3.7) \quad \begin{aligned} x^{m+1}(\Psi_a(q)) &= x^{m+1}(\Psi_a(\Psi_b(\hat{q}))) \\ &= x^{m+1}(\Psi_{a+b}(\hat{q})) \\ &= a + b \quad (\text{by (1) of (3.5)}) \\ &= a + x^{m+1}(\Psi_b(\hat{q})) \quad (\text{by (1) of (3.5)}) \\ &= a + x^{m+1}(q). \end{aligned}$$

We have proved (1). Moreover, we conclude (2) by (b) of Definition 1.1 and (1). \square

Proof. (of Theorem C') Let $\phi := u(x^1, x^2, \dots, x^m) - x^{m+1}$ and $\psi := v(x^1, x^2, \dots, x^m) - x^{m+1}$. Observe that $v \geq u$ implies

$$(3.8) \quad \psi \geq \phi.$$

For any q near p_0 , there are $a = x^{m+1}(q) - v(x_0^1, \dots, x_0^m)$, $b = x^{m+1}(q) - u(x_0^1, \dots, x_0^m) \in (-\delta, \delta)$ in which $x_0^1 = x^1(q)$, \dots , $x_0^m = x^m(q)$, such that

$$(3.9) \quad \begin{aligned} q &= \Psi_a(x_0^1, \dots, x_0^m, v(x_0^1, \dots, x_0^m)) \\ &= \Psi_b(x_0^1, \dots, x_0^m, u(x_0^1, \dots, x_0^m)). \end{aligned}$$

by transversality of isometric translations. Note that Ψ_a is compatible with ϕ and ψ . From (3.9) we compute

$$(3.10) \quad \begin{aligned} H_{\psi,p}(q) &= H_{\psi,p}(\Psi_a(x_0^1, \dots, x_0^m, v(x_0^1, \dots, x_0^m))) \\ &= H_{\psi,p}(x_0^1, \dots, x_0^m, v(x_0^1, \dots, x_0^m)) \quad (\text{by Proposition 3.1}) \\ &= H_p(v)(x_0^1, \dots, x_0^m) \quad (\text{by (1.9)}) \\ &\leq H_p(u)(x_0^1, \dots, x_0^m) \quad (\text{by assumption (2)}) \\ &= H_{\phi,p}(x_0^1, \dots, x_0^m, u(x_0^1, \dots, x_0^m)) \quad (\text{by (1.9)}) \\ &= H_{\phi,p}(\Psi_b(x_0^1, \dots, x_0^m, u(x_0^1, \dots, x_0^m))) \quad (\text{by Proposition 3.1}) \\ &= H_{\phi,p}(q) \quad (\text{by (3.9)}). \end{aligned}$$

In view of (3.8) and (3.10), we have $\psi = \phi$ on $\Sigma_1 := \{\phi = 0\}$ by Theorem B (a). It follows that $v \equiv u$ in a neighborhood of $p_0 \in \Sigma$. □

The argument in (3.10) shows the following fact.

Corollary 3.2. *Suppose $(M, \langle \cdot, \cdot \rangle, dv_M)$ of dimension $m+1$ with $m \geq 3$ has isometric translations Ψ_a near $p \in M$, transversal to a hypersurface Σ passing through p . Take a system of translation-isometric coordinates x^1, x^2, \dots, x^{m+1} in a neighborhood V of p such that $x^{m+1} = 0$ on Σ . Suppose $u : \Sigma \cap V \rightarrow \mathbb{R}$ defines a graph $\{(x^1, x^2, \dots, x^m, u(x^1, x^2, \dots, x^m))\} \subset V$. Let $\phi(x^1, x^2, \dots, x^m, x^{m+1}) := u(x^1, x^2, \dots, x^m) - x^{m+1}$. Then $H_\phi(q) = H(u)(\pi_\Sigma(q))$ wherever defined, in which π_Σ is the projection to Σ (i.e., $(x^1, x^2, \dots, x^m, x^{m+1}) \rightarrow (x^1, x^2, \dots, x^m)$ in translation-isometric coordinates)*

We now want to apply our theory of translation-isometric coordinates to the situation of l_a graphs. Observe that $\eta^2 = x^2$, $\eta^3 = x^3, \dots$, $\eta^{2n} = x^{2n}$, and $\tau := z + x^1 x^{n+1}$ are invariant under l_a , Heisenberg translations in the direction $\frac{\partial}{\partial x^1}$. So an l_a graph u is a hypersurface $(u(\eta^2, \eta^3, \dots, \eta^{2n}, \tau), 0, \dots, 0) \circ (0, \eta^2, \dots, \eta^{2n}, \tau)$ of H_n , parametrized by $\eta^2, \eta^3, \dots, \eta^{2n}, \tau$ so that

$$(3.11) \quad \begin{aligned} x^1 &= u(\eta^2, \eta^3, \dots, \eta^{2n}, \tau), \\ x^2 &= \eta^2, \dots, x^{2n} = \eta^{2n}, \\ z &= \tau - \eta^{n+1} u(\eta^2, \eta^3, \dots, \eta^{2n}, \tau). \end{aligned}$$

In coordinates $\eta^2, \eta^3, \dots, \eta^{2n}, \tau$, and x^1 , we write the standard contact form Θ for H_n as follows:

$$(3.12) \quad \begin{aligned} \Theta &: = dz + x^1 dx^{n+1} - x^{n+1} dx^1 + \sum_{j=2}^n (x^j dx^{n+j} - x^{n+j} dx^j) \\ &= d\tau - 2\eta^{n+1} dx^1 + \sum_{j=2}^n (\eta^j d\eta^{n+j} - \eta^{n+j} d\eta^j). \end{aligned}$$

Observe that

$$\begin{aligned} \dot{e}_2 &: = \frac{\partial}{\partial \eta^2} + \eta^{n+2} \frac{\partial}{\partial \tau}, \dots, \dot{e}_n := \frac{\partial}{\partial \eta^n} + \eta^{2n} \frac{\partial}{\partial \tau}, \\ \dot{e}_1 &: = \frac{\partial}{\partial x^1} + 2\eta^{n+1} \frac{\partial}{\partial \tau}, \dot{e}_{n+1} := \frac{\partial}{\partial \eta^{n+1}}, \\ \dot{e}_{n+2} &: = \frac{\partial}{\partial \eta^{n+2}} - \eta^2 \frac{\partial}{\partial \tau}, \dots, \dot{e}_{2n} := \frac{\partial}{\partial \eta^{2n}} - \eta^n \frac{\partial}{\partial \tau}. \end{aligned}$$

form an orthonormal basis of $\ker \Theta$ with respect to the Levi metric $\frac{1}{2}d\Theta(\cdot, J\cdot)$ (see Subsection B in the Appendix for more explanation). We remark that the above \dot{e}_j , \dot{e}_{n+j} are the same as the vectors \hat{e}_j , $\hat{e}_{j'}$, resp. in Subsection B of the Appendix, but expressed in different coordinates. Let $\phi := u(\eta^2, \eta^3, \dots, \eta^{2n}, \tau) - x^1$. We compute

$$(3.13) \quad \begin{aligned} \dot{e}_1 \phi &= -1 + 2\eta^{n+1} \partial_\tau u, \quad \dot{e}_{n+1} \phi = \partial_{\eta^{n+1}} u, \\ \dot{e}_2 \phi &= \partial_{\eta^2} u + \eta^{n+2} \partial_\tau u, \dots, \dot{e}_{2n} \phi = \partial_{\eta^{2n}} u - \eta^n \partial_\tau u. \end{aligned}$$

Hence $d\phi = (\dot{e}_1 \phi) dx^1 + \sum_{j=2}^{2n} (\dot{e}_j \phi) d\eta^j + (\partial_\tau \phi) \Theta$ has the length with respect to the subriemannian metric (7.6) as follows:

$$(3.14) \quad \begin{aligned} |d\phi|_*^2 &= \sum_{j=1}^{2n} (\dot{e}_j \phi)^2 = |W(u - x^1)|^2 \\ &= (1 - 2\eta^{n+1} \partial_\tau u)^2 + (\partial_{\eta^{n+1}} u)^2 \\ &\quad + \sum_{j=2}^n [(\partial_{\eta^j} u + \eta^{n+j} \partial_\tau u)^2 + (\partial_{\eta^{n+j}} u - \eta^j \partial_\tau u)^2] \\ &= 1 - 4\eta^{n+1} \partial_\tau u + |Wu|^2 \end{aligned}$$

where W is the vector-valued operator $(\dot{e}_1, \dot{e}_2, \dots, \dot{e}_{2n})$. Note that the standard volume form dV_{H_n} of H_n equals $dx^1 \wedge d\eta^2 \wedge \dots \wedge d\eta^{2n} \wedge \Theta$.

From (7.8) we compute H_ϕ as follows:

$$\begin{aligned} &d\left(\frac{d\phi}{|d\phi|_*} \lrcorner dV_{H_n}\right) \\ &= d\left\{\left[\frac{\dot{e}_1 \phi}{|d\phi|_*} dx^1 + \sum_{j=2}^{2n} \frac{\dot{e}_j \phi}{|d\phi|_*} d\eta^j + \frac{\partial_\tau \phi}{|d\phi|_*} \Theta\right] \lrcorner dV_{H_n}\right\} \\ &= \left[\dot{e}_1\left(\frac{\dot{e}_1 \phi}{|d\phi|_*}\right) + \sum_{j=2}^{2n} \dot{e}_j\left(\frac{\dot{e}_j \phi}{|d\phi|_*}\right)\right] dV_{H_n} \end{aligned}$$

So we have

$$\begin{aligned}
H_\phi &= \dot{e}_1\left(\frac{\dot{e}_1\phi}{|d\phi|_*}\right) + \sum_{j=2}^{2n} \dot{e}_j\left(\frac{\dot{e}_j\phi}{|d\phi|_*}\right) = W \cdot \left(\frac{W(u-x^1)}{|W(u-x^1)|} \right) \\
&= 2\eta^{n+1}\partial_\tau \left(\frac{-1+2\eta^{n+1}\partial_\tau u}{\sqrt{1-4\eta^{n+1}\partial_\tau u + |Wu|^2}} \right) \\
&\quad + \partial_{\eta^{n+1}} \left(\frac{\partial_{\eta^{n+1}} u}{\sqrt{1-4\eta^{n+1}\partial_\tau u + |Wu|^2}} \right) \\
&\quad + \sum_{j=2}^n [(\partial_{\eta^j} + \eta^{n+j}\partial_\tau) \left(\frac{\partial_{\eta^j} u + \eta^{n+j}\partial_\tau u}{\sqrt{1-4\eta^{n+1}\partial_\tau u + |Wu|^2}} \right) \\
&\quad + (\partial_{\eta^{n+j}} - \eta^j\partial_\tau) \left(\frac{\partial_{\eta^{n+j}} u - \eta^j\partial_\tau u}{\sqrt{1-4\eta^{n+1}\partial_\tau u + |Wu|^2}} \right)].
\end{aligned}$$

So $H(u)$ has the above expression (see (1.9) with $p = 0$ for the definition). Let Σ_1 denote the hypersurface defined by $u(\eta^2, \eta^3, \dots, \eta^{2n}, \tau) = x^1$. As in the proof of Corollary D, we observe that $\dim \xi \cap T\Sigma_1 = 2n-1 \geq 3$ for $n \geq 2$. We can then find a J -invariant pair of nonzero vectors X, JX in $\xi \cap T\Sigma_1$. The Lie bracket $[X, JX]$ generates a direction out of contact distribution ξ . Therefore $\text{rank}(\mathcal{L}(\xi \cap T\Sigma_1)) = 2n = m$ near p_0 . By Theorem C' we have

Corollary D'. *Suppose $n \geq 2$. Let $x^1 = u(\eta^2, \eta^3, \dots, \eta^{2n}, \tau)$ and $x^1 = v(\eta^2, \eta^3, \dots, \eta^{2n}, \tau)$ be two l_a graphs defined on a common domain Ω . Assume $v = u$ at $p_0 \in \Omega$, $v \geq u$ in Ω , and $H(v) \leq H(u)$ in Ω . Then $v \equiv u$ near p_0 .*

We now want to apply our theory of translation-isometric coordinates to the situation of intrinsic graphs. Let us recall ([1]) that an intrinsic graph u is a hypersurface of H_n , parametrized by $\eta^2, \eta^3, \dots, \eta^{2n}, \tau$ so that $(x^1, \dots, x^{2n}, z) = (0, \eta^2, \eta^3, \dots, \eta^{2n}, \tau) \cdot (u(\eta^2, \eta^3, \dots, \eta^{2n}, \tau), 0, \dots, 0)$ or

$$\begin{aligned}
(3.15) \quad x^1 &= u(\eta^2, \eta^3, \dots, \eta^{2n}, \tau), \\
x^2 &= \eta^2, \dots, x^{2n} = \eta^{2n}, \\
z &= \tau + \eta^{n+1}u(\eta^2, \eta^3, \dots, \eta^{2n}, \tau).
\end{aligned}$$

In coordinates $\eta^2, \eta^3, \dots, \eta^{2n}, \tau$, and x^1 , we write the standard contact form Θ for H_n as follows:

$$\begin{aligned}
(3.16) \quad \Theta &: = dz + x^1 dx^{n+1} - x^{n+1} dx^1 + \sum_{j=2}^n (x^j dx^{n+j} - x^{n+j} dx^j) \\
&= d\tau + 2x^1 d\eta^{n+1} + \sum_{j=2}^n (\eta^j d\eta^{n+j} - \eta^{n+j} d\eta^j).
\end{aligned}$$

Observe that

$$\begin{aligned}\dot{e}_1 &:= \frac{\partial}{\partial x^1}, \quad \dot{e}_2 := \frac{\partial}{\partial \eta^2} + \eta^{n+2} \frac{\partial}{\partial \tau}, \dots, \dot{e}_n := \frac{\partial}{\partial \eta^n} + \eta^{2n} \frac{\partial}{\partial \tau}, \\ \dot{e}_{n+1} &:= \frac{\partial}{\partial \eta^{n+1}} - 2x^1 \frac{\partial}{\partial \tau}, \\ \dot{e}_{n+2} &:= \frac{\partial}{\partial \eta^{n+2}} - \eta^2 \frac{\partial}{\partial \tau}, \dots, \dot{e}_{2n} := \frac{\partial}{\partial \eta^{2n}} - \eta^n \frac{\partial}{\partial \tau}\end{aligned}$$

form an orthonormal basis of $\ker \Theta$ with respect to the Levi metric $\frac{1}{2}d\Theta(\cdot, J\cdot)$ (see Subsection B in the Appendix for more explanation). Let $\phi := u(\eta^2, \eta^3, \dots, \eta^{2n}, \tau) - x^1$. We compute

$$\begin{aligned}(3.17) \quad \dot{e}_1 \phi &= -1, \dot{e}_2 \phi = \partial_{\eta^2} u + \eta^{n+2} \partial_{\tau} u, \dots, \\ \dot{e}_{n+1} \phi &= \partial_{\eta^{n+1}} u - 2x^1 \partial_{\tau} u, \dots, \dot{e}_{2n} \phi = \partial_{\eta^{2n}} u - \eta^n \partial_{\tau} u.\end{aligned}$$

Hence $d\phi = (\dot{e}_1 \phi) dx^1 + \sum_{j=2}^{2n} (\dot{e}_j \phi) d\eta^j + (\partial_{\tau} \phi) \Theta$ has the length with respect to the subriemannian metric (7.6) as follows:

$$(3.18) \quad |d\phi|_* = \sqrt{\sum_{j=1}^{2n} (\dot{e}_j \phi)^2} = \sqrt{1 + |W^u(u)|^2}$$

on the graph described by $x^1 = u(\eta^2, \eta^3, \dots, \eta^{2n}, \tau)$. Here $W^u := (\dot{e}_2, \dots, \dot{e}_n, \dot{e}_{n+1}^u, \dot{e}_{n+2}, \dots, \dot{e}_{2n})$ and $\dot{e}_{n+1}^u := \frac{\partial}{\partial \eta^{n+1}} - 2u \frac{\partial}{\partial \tau}$. Note that the standard volume form dV_{H_n} of H_n equals $dx^1 \wedge d\eta^2 \wedge \dots \wedge d\eta^{2n} \wedge \Theta$. On $\Sigma := \{\phi = 0\}$ we have $0 = d\phi = (\dot{e}_1 \phi) dx^1 + \sum_{j=2}^{2n} (\dot{e}_j \phi) d\eta^j + (\partial_{\tau} \phi) \Theta$. So by (3.17) we get

$$(3.19) \quad dx^1 = \sum_{j=2}^{2n} (\dot{e}_j \phi) d\eta^j + (\partial_{\tau} \phi) \Theta.$$

Now we compute the area (or volume) element dv_{ϕ} for the hypersurface $\{\phi = \phi(p_0)\}$:

$$\begin{aligned}(3.20) \quad dv_{\phi} &:= \frac{d\phi}{|d\phi|_*} \lrcorner dV_{H_n} \\ &= \frac{1}{|d\phi|_*} [(\dot{e}_1 \phi) d\eta^2 \wedge \dots \wedge d\eta^{2n} \wedge \Theta \\ &\quad + (-1)^{j-1} \sum_{j=2}^{2n} (\dot{e}_j \phi) dx^1 \wedge d\eta^2 \wedge \dots \wedge d\hat{\eta}^j \wedge d\eta^{2n} \wedge \Theta] \\ &= \frac{1}{|d\phi|_*} [-1 - \sum_{j=2}^{2n} (\dot{e}_j \phi)^2] d\eta^2 \wedge \dots \wedge d\eta^{2n} \wedge \Theta \quad (\text{by (3.19)}) \\ &= -|d\phi|_* d\eta^2 \wedge \dots \wedge d\eta^{2n} \wedge d\tau \\ &= -\sqrt{1 + |W^u(u)|^2} d\eta^2 \wedge \dots \wedge d\eta^{2n} \wedge d\tau\end{aligned}$$

by (3.16) and (3.18). From (7.8) we compute H_ϕ as follows:

$$\begin{aligned}
& d\left(\frac{d\phi}{|d\phi|_*} \lrcorner dV_{H_n}\right) \\
&= d\left\{\left[\frac{\dot{e}_1\phi}{|d\phi|_*}dx^1 + \sum_{j=2}^{2n} \frac{\dot{e}_j\phi}{|d\phi|_*}d\eta^j + \frac{\partial_\tau\phi}{|d\phi|_*}\Theta\right] \lrcorner dV_{H_n}\right\} \\
&= \left[\dot{e}_1\left(\frac{\dot{e}_1\phi}{|d\phi|_*}\right) + \sum_{j=2}^{2n} \dot{e}_j\left(\frac{\dot{e}_j\phi}{|d\phi|_*}\right)\right]dV_{H_n} \\
&= \sum_{j=2}^{2n} \dot{e}_j\left(\frac{\dot{e}_j\phi}{|d\phi|_*}\right)dV_{H_n}
\end{aligned}$$

since $|d\phi|_*$ is independent of x^1 by (3.18) and hence $\dot{e}_1\left(\frac{\dot{e}_1\phi}{|d\phi|_*}\right) = \partial_{x^1}\left(\frac{-1}{|d\phi|_*}\right) = 0$. So we have

$$\begin{aligned}
(3.21) \quad H_\phi &= \sum_{j=2}^{2n} \dot{e}_j\left(\frac{\dot{e}_j\phi}{|d\phi|_*}\right) \\
&= W^u \cdot \left(\frac{W^u(u)}{\sqrt{1+|W^u(u)|^2}}\right)
\end{aligned}$$

at $(\eta^2, \eta^3, \dots, \eta^{2n}, \tau, u(\eta^2, \eta^3, \dots, \eta^{2n}, \tau))$ (cf. (1.11)).

Proof. (of Theorem E) We observe that an intrinsic graph is congruent with an l_a graph by a rotation. Suppose we have an intrinsic graph described by (3.15). Define a rotation $\Psi : (x^1, \dots, x^{2n}, z) \in H_n \rightarrow (\tilde{x}^1, \dots, \tilde{x}^{2n}, \tilde{z}) \in H_n$ by

$$\begin{aligned}
(3.22) \quad \tilde{x}^1 &= x^{n+1}, \tilde{x}^{n+1} = -x^1, \\
\tilde{x}^j &= x^j, 1 \leq j \leq 2n, j \neq 1, j \neq n+1, \\
\tilde{z} &= z.
\end{aligned}$$

From $u(\eta^2, \eta^3, \dots, \eta^{2n}, \tau) - x^1 = 0$ and $u_{\eta^{n+1}} \neq 0$ at p_0 , we can express $\eta^{n+1} = \eta^{n+1}(x^1, \eta^2, \dots, \hat{\eta}^{n+1}, \dots, \eta^{2n}, \tau)$ near $(u(p_0), \eta_0^2, \dots, \hat{\eta}_0^{n+1}, \dots, \eta_0^{2n}, \tau_0)$ by implicit function theorem. Here $\hat{\eta}^{n+1}$ means η^{n+1} deleted. In view of (3.22) and (3.15), we get

$$\begin{aligned}
(3.23) \quad \tilde{x}^1 &= \eta^{n+1}(x^1, \eta^2, \dots, \hat{\eta}^{n+1}, \dots, \eta^{2n}, \tau), \\
\tilde{x}^j &= \eta^j, 2 \leq j \leq 2n, j \neq n+1, \\
\tilde{x}^{n+1} &= -x^1, \\
\tilde{z} &= z = \tau + x^1 \eta^{n+1}(x^1, \eta^2, \dots, \hat{\eta}^{n+1}, \dots, \eta^{2n}, \tau).
\end{aligned}$$

Let $\zeta = -x^1$ and

$$\begin{aligned}
(3.24) \quad & \tilde{\eta}^{n+1}(\eta^2, \dots, \zeta, \dots, \eta^{2n}, \tau) \text{ } (\hat{\eta}^{n+1} \text{ is replaced by } \zeta) \\
& : = \eta^{n+1}(x^1, \eta^2, \dots, \hat{\eta}^{n+1}, \dots, \eta^{2n}, \tau).
\end{aligned}$$

It follows from (3.23) and (3.24) that the image of an intrinsic graph under the rotation Ψ can be depicted as $(\tilde{\eta}^{n+1}(\eta^2, \dots, \zeta, \dots, \eta^{2n}, \tau), 0, \dots, 0) \circ (0, \eta^2, \dots, \zeta, \dots, \eta^{2n}, \tau)$, an l_a graph. Let $\tilde{\xi}^{n+1}$ denote the l_a graph corresponding to the intrinsic graph v under the rotation Ψ . Near p_0 the condition $v \geq u$ implies $\tilde{\xi}^{n+1} \geq \tilde{\eta}^{n+1}$ ($\tilde{\xi}^{n+1} \leq$

$\tilde{\eta}^{n+1}$, resp.) if at p_0 , $\nu_{\eta^{n+1}} = u_{\eta^{n+1}} < 0$ (> 0 , resp.). On the other hand, $H_{\xi^{n+1}}$ ($H_{\tilde{\eta}^{n+1}}$, resp.) is the same as H_v (H_u , resp.) at the same point in the graph for the case of $\nu_{\eta^{n+1}} = u_{\eta^{n+1}} < 0$ at p_0 . Therefore the condition $H_v \leq H_u$ is reduced to $H_{\xi^{n+1}} \leq H_{\tilde{\eta}^{n+1}}$ in some corresponding small neighborhood when either one is constant. For the case of $\nu_{\eta^{n+1}} = u_{\eta^{n+1}} > 0$ at p_0 , $H_{\xi^{n+1}}$ ($H_{\tilde{\eta}^{n+1}}$, resp.) is the same as $-H_v$ ($-H_u$, resp.) at the same point in the graph. So $H_v \leq H_u$ is equivalent to $H_{\xi^{n+1}} \geq H_{\tilde{\eta}^{n+1}}$ in some corresponding small neighborhood when either one is constant. Now the theorem follows from Corollary D'. \square

We remark that if both v and u do not have constant horizontal (or p -)mean curvature, then we won't be able to reduce $H_v \leq H_u$ to $H_{\xi^{n+1}} \leq$ (or \geq) $H_{\tilde{\eta}^{n+1}}$ in general. The reason is that we are comparing horizontal (or p -)mean curvature at different pairs of points on two hypersurfaces.

4. Proofs of Theorem F, Corollaries G, H, and I

Proof. (of Theorem F) We want to show that $H_{\bar{F}}(u)$ at (x^1, \dots, x^m) is the same as H_ϕ for $\phi = u(x^1, \dots, x^m) - x^{m+1}$ at $(x^1, \dots, x^m, u(x^1, \dots, x^m))$ with respect to a certain subriemannian structure on H_n . Write $\vec{F} = (F_1, \dots, F_m)$. Let

$$(4.1) \quad \omega^{m+1} = dx^{m+1} + \sum_{j=1}^m F_j dx^j.$$

Define subriemannian metric $\langle \cdot, \cdot \rangle_F$ on T^*H_n by

$$\begin{aligned} \langle dx^i, dx^j \rangle_F &= \delta_{ij}, \quad 1 \leq i, j \leq m, \\ \langle \eta, \omega^{m+1} \rangle_F &= 0 \end{aligned}$$

for any one form η . The vectors dual to $dx^1, \dots, dx^m, \omega^{m+1}$ read

$$\begin{aligned} e_j^F &= \frac{\partial}{\partial x^j} - F_j \frac{\partial}{\partial x^{m+1}}, \quad 1 \leq j \leq m \\ e_{m+1} &= \frac{\partial}{\partial x^{m+1}}. \end{aligned}$$

Take the standard volume form $dV_{H_n} := dx^1 \wedge \dots \wedge dx^m \wedge \omega^{m+1} = dx^1 \wedge \dots \wedge dx^m \wedge dx^{m+1}$. We compute H_ϕ as follows:

$$\begin{aligned} (4.2) \quad & d\left(\frac{d\phi}{|d\phi|_{\langle \cdot, \cdot \rangle_F}} \lrcorner \langle \cdot, \cdot \rangle_F dV_{H_n}\right) \\ &= \sum_{j=1}^m e_j^F \left(\frac{e_j^F(\phi)}{|d\phi|_{\langle \cdot, \cdot \rangle_F}} \right) dV_{H_n} \end{aligned}$$

where

$$|d\phi|_{\langle \cdot, \cdot \rangle_F} = \left(\sum_{j=1}^m (e_j^F(\phi))^2 \right)^{1/2}.$$

From (4.2) we have

$$(4.3) \quad H_\phi = \sum_{j=1}^m e_j^F \left(\frac{e_j^F(\phi)}{|d\phi|_{\langle \cdot, \cdot \rangle_F}} \right).$$

On the other hand, we compute, for $\phi = u(x^1, \dots, x^m) - x^{m+1}$,

$$e_j^F(\phi) = \frac{\partial u}{\partial x^j} + F_j.$$

Hence we have

$$|d\phi|_{\langle \cdot, \cdot \rangle_F} = \left(\sum_{j=1}^m \left(\frac{\partial u}{\partial x^j} + F_j \right)^2 \right)^{1/2} = |\nabla u + \vec{F}|_{R^m}.$$

Observe that $\frac{e_j^F(\phi)}{|d\phi|_{\langle \cdot, \cdot \rangle_F}}$ is independent of x^{m+1} . It follows that

$$(4.4) \quad \sum_{j=1}^m e_j^F \left(\frac{e_j^F(\phi)}{|d\phi|_{\langle \cdot, \cdot \rangle_F}} \right) = \operatorname{div} \left(\frac{\nabla u + \vec{F}}{|\nabla u + \vec{F}|_{R^m}} \right).$$

From (4.3), (4.4), we have shown $H_{\vec{F}}(u)$ at (x^1, \dots, x^m) equals H_ϕ at $(x^1, \dots, x^m, u(x^1, \dots, x^m))$. It is obvious that the translation along x^{m+1} preserves $\langle \cdot, \cdot \rangle_F$ and dV_{H_n} . Moreover, x^1, \dots, x^m, x^{m+1} are translation-isometric coordinates with respect to the $x^1 \dots x^m$ hyperplane. Observe that $N_1^\perp(u), N_2^\perp(u), \dots, N_{m-1}^\perp(u)$ are the $x^1 \dots x^m$ hyperplane projection of (a choice of) X_1, \dots, X_{m-1} in (2.10), resp.. The conclusion follows from Theorem C' for $l = 1$. \square

Proof. (of Corollary G) Let

$$\begin{aligned} \Theta_v &: = dv + F_j dx^j \\ &= \sum_{j=1}^m (v_j + F_j) dx^j. \end{aligned}$$

Observe that $\Theta_v(N_\gamma^\perp(v)) = 0$ since $N_\alpha^\perp(v)$ is perpendicular to $\nabla v + \vec{F}$. It follows that

$$\begin{aligned} (4.5) \quad -\Theta_v([N_\alpha^\perp(v), N_\beta^\perp(v)]) &= d\Theta_v(N_\alpha^\perp(v), N_\beta^\perp(v)) \\ &= \sum_{k,j=1}^m (\partial_k F_j) dx^k \wedge dx^j (N_\alpha^\perp(v), N_\beta^\perp(v)) \\ &= \sum_{k,j=1}^m (\partial_k F_j) (b_\alpha^k b_\beta^j - b_\alpha^j b_\beta^k) \\ &= \sum_{k < j} (\partial_k F_j - \partial_j F_k) (b_\alpha^k b_\beta^j - b_\alpha^j b_\beta^k) \end{aligned}$$

by (1.13). By assumption we get $\Theta_v([N_\alpha^\perp(v), N_\beta^\perp(v)]) \neq 0$ in view of (4.5). So $[N_\alpha^\perp(v), N_\beta^\perp(v)]$ is nonzero and transversal to the kernel of Θ_v , the space spanned by all $N_\gamma^\perp(v), \gamma = 1, 2, \dots, m-1$. We then have that the rank of $\mathcal{L}(N_1^\perp(v), N_2^\perp(v), \dots, N_{m-1}^\perp(v))$ is constant m in U . Now the conclusion follows from Theorem F. \square

We now want to find an intrinsic criterion for the Lie span condition $\operatorname{rank}(\mathcal{L}(N_1^\perp(u), N_2^\perp(u), \dots, N_{m-1}^\perp(u))) = m$ in Theorem F or more generally $\operatorname{rank}(\mathcal{L}(\xi \cap T\Sigma_1)) = m$ in Theorem C to hold. Suppose Σ is a hypersurface in a subriemannian manifold

$(M, \langle \cdot, \cdot \rangle^*)$ of dimension $m+1$. Define $G : T^*M \rightarrow TM$ by $\omega(G(\eta)) = \langle \omega, \eta \rangle^*$ for $\omega, \eta \in T^*M$. Let $\xi := \text{Range}(G) \subset TM$. We assume (as always)

$$\dim \xi = \text{constant } m+1-l$$

where $l = \dim \ker G$. We wonder when

$$\mathcal{L}(\xi \cap T\Sigma) = \mathcal{L}(X_1, \dots, X_{m-l}),$$

where X_1, \dots, X_{m-l} form a basis of local sections of $\xi \cap T\Sigma$ (near p_0 where $\xi \neq T\Sigma$), has rank m . Let us start with the case $l = 1$. Take a nonzero 1-form $\theta \in \ker G$. It is easy to see that θ annihilates vectors in ξ . So

$$(4.6) \quad \theta(X_j) = 0$$

for $1 \leq j \leq m-1$. We want Lie bracket of a pair of X_j to generate a direction not in ξ (which will imply $\mathcal{L}(X_1, \dots, X_{m-1})$ has rank m). Suppose the converse holds. Then we have

$$(4.7) \quad \begin{aligned} d\theta(X_i, X_j) &= X_i(\theta(X_j)) - X_j(\theta(X_i)) - \theta([X_i, X_j]) \\ &= -\theta([X_i, X_j]) = 0 \end{aligned}$$

by (4.6) for $1 \leq j \leq m-1$. It follows from (4.7) that $d\theta|_{\xi \times \xi}$, the bilinear form $d\theta$ restricted to $\xi \times \xi$, has rank ≤ 2 . So we have proved the following result.

Proposition 4.1. *Let Σ be a hypersurface in a subriemannian manifold $(M, \langle \cdot, \cdot \rangle^*)$ of dimension $m+1$. Suppose $\dim \xi = \text{constant } m$ (i.e., $l = 1$). Assume*

$$(4.8) \quad \text{rank}(d\theta|_{\xi \times \xi}) \geq 3.$$

Then there holds (1.7) and hence we have

$$\text{rank}(\mathcal{L}(\xi \cap T\Sigma)) = m.$$

We remark that condition (4.8) is independent of the choice of θ . Since $\dim \ker G = 1$, another nonzero choice $\tilde{\theta} \in \ker G$ will be a nonzero multiple of θ . That is, $\tilde{\theta} = \lambda\theta$ with $\lambda \neq 0$. It follows that

$$\begin{aligned} d\tilde{\theta}|_{\xi \times \xi} &= \lambda d\theta|_{\xi \times \xi} + (d\lambda \wedge \theta)|_{\xi \times \xi} \\ &= \lambda d\theta|_{\xi \times \xi}. \end{aligned}$$

So $d\tilde{\theta}|_{\xi \times \xi}$ has the same rank as $d\theta|_{\xi \times \xi}$.

Proof. (of Corollary H) From the proof of Theorem F, we learn that $H_{\bar{F}}(u)$ at (x^1, \dots, x^m) is the same as H_ϕ for $\phi = u(x^1, \dots, x^m) - x^{m+1}$ at $(x^1, \dots, x^m, u(x^1, \dots, x^m))$ with respect to a certain subriemannian structure on H_n with $m = 2n$. This subriemannian structure has the property that $\dim \xi = \text{constant } m$ (i.e., $l = 1$). We take

$$\theta = \omega^{m+1} = dx^{m+1} + \sum_{j=1}^m F_j dx^j$$

(see (4.1)). Compute

$$d\theta = \sum_{k < j} (\partial_k F_j - \partial_j F_k) dx^k \wedge dx^j.$$

Therefore condition (1.15) means $\text{rank}(d\theta|_{\xi \times \xi}) \geq 3$. Note that $N_1^\perp(v)$, $N_2^\perp(v)$, ..., $N_{m-1}^\perp(v)$ are the $x^1 \dots x^m$ hyperplane projection of (a choice of) X_1, \dots, X_{m-1} , resp. in (2.10) for $\phi = v - x^{m+1}$. The conclusion follows from Proposition 4.1 and Theorem F. \square

Proof. (of Corollary I) For such an $\vec{F} = (-x^2, x^1, \dots, -x^{2n}, x^{2n-1})$, we compute

$$(4.9) \quad \begin{aligned} \partial_{2k-1} F_{2k} - \partial_{2k} F_{2k-1} &= 2 \text{ for } k = 1, \dots, n; \\ \partial_k F_j - \partial_j F_k &= 0 \text{ (} k < j \text{) otherwise.} \end{aligned}$$

So the rank of the matrix $(\partial_k F_j - \partial_j F_k)$ equals $2n$ by (4.9). By assumption we have $m = 2n \geq 4 > 3$. The conclusion follows from Corollary H. \square

We are going to give a sufficient condition for the rank estimate in the case of general l . Let $\theta^1, \dots, \theta^l$ be a basis of $\ker G$. Choose an (codimension 1, resp.) orthonormal basis X_1, \dots, X_{m+1-l} (X_1, \dots, X_{m-l} , resp.) of ξ and its dual forms $\omega^1, \dots, \omega^{m+1-l}$ ($\omega^1, \dots, \omega^{m-l}$, resp.) (not unique). Then we can find unique vector fields T_1, \dots, T_l such that $\{X_1, \dots, X_{m+1-l}, T_1, \dots, T_l\}$ is dual to $\{\omega^1, \dots, \omega^{m+1-l}, \theta^1, \dots, \theta^l\}$. Write

$$(4.10) \quad [X_i, X_j] = \sum_{\alpha=1}^l W_{ij}^\alpha T_\alpha \text{ mod } \xi$$

for $1 \leq i, j \leq m+1-l$ ($1 \leq i, j \leq m-l$, resp.). For fixed α we view (W_{ij}^α) as an $(m+1-l) \times (m+1-l)$ matrix. Since it is skew-symmetric, it has $\frac{(m+1-l)(m-l)}{2}$ degrees of freedom.

Proposition 4.2. *Suppose*

$$(4.11) \quad \begin{aligned} \frac{(m+1-l)(m-l)}{2} &\geq l \\ \left(\frac{(m-l)(m-l-1)}{2} \right) &\geq l, \text{ resp.} \end{aligned}$$

Then $\text{rank}(\mathcal{L}(\xi)) = m+1$ ($\text{rank}(\mathcal{L}(X_1, \dots, X_{m-l})) \geq m$, resp.) if $d\theta^\alpha \lrcorner \omega^1 \wedge \dots \wedge \omega^{m+1-l}$ ($\omega^1 \wedge \dots \wedge \omega^{m-l}$, resp.), $\alpha = 1, \dots, l$, are linearly independent, or equivalently $[W_{ij}^\alpha]$ as $(m+1-l) \times (m+1-l)$ matrix ($(m-l) \times (m-l)$ matrix, resp.), $\alpha = 1, \dots, l$, are linearly independent.

Proof. In view of (4.10) and (4.11), the linear independence of $[W_{ij}^\alpha]$, $\alpha = 1, \dots, l$, implies $T_1, \dots, T_l \in \mathcal{L}(\xi)$ ($\mathcal{L}(X_1, \dots, X_{m-l})$, resp.) by basic linear algebra. It follows that $\text{rank}(\mathcal{L}(\xi)) \geq (m+1-l) + l = m+1$ ($\text{rank}(\mathcal{L}(X_1, \dots, X_{m-l})) \geq (m-l) + l = m$, resp.). On the other hand, clearly we have $\text{rank}(\mathcal{L}(\xi)) \leq m+1$. By the formula

$$d\theta^\alpha(X_i, X_j) = X_i(\theta^\alpha(X_j)) - X_j(\theta^\alpha(X_i)) - \theta^\alpha([X_i, X_j]),$$

we obtain

$$d\theta^\alpha = - \sum_{i < j} W_{ij}^\alpha \omega^i \wedge \omega^j \mod \theta^1, \dots, \theta^l$$

by (4.10). We then learn that two conditions are equivalent. \square

5. SINGULARITIES AND PROOFS OF THEOREMS C'' , J, K, \tilde{C}'' , J'

Proof. (of Theorem C'') By Theorem 5.2' in [12], we have $N_{\vec{F}}(v) = N_{\vec{F}}(u)$ in $\Omega^+ \setminus \{S_{\vec{F}}(u) \cup S_{\vec{F}}(v)\}$ where $\Omega^+ := \{p \in \Omega \mid u(p) - v(p) > 0\}$. Suppose Ω^+ is not empty. Then applying Lemma 5.3' in [12] with $\Omega = \Omega^+$ and \vec{F}^* replaced by \vec{F}^b in the proof, we obtain $v = u$ in Ω^+ , a contradiction. We have proved $v \geq u$ in Ω . \square

Proof. (of Theorem J) Case 1: Suppose $v = u$ at $q \neq p_0$. Since p_0 is the only singular point of u (v , resp.), q is nonsingular with respect to u (v , resp.). On the other hand, that $v - u \geq 0$ and $v - u = 0$ at q implies that $\nabla(v - u) = 0$ at q . It follows that at q ,

$$\begin{aligned} \nabla v + \vec{F} &= \nabla u + \vec{F} \neq 0 \\ (\nabla u + \vec{F} &= \nabla v + \vec{F} \neq 0, \text{ resp.}) \end{aligned}$$

since q is nonsingular with respect to u (v , resp.). So q is also nonsingular with respect to v (u , resp.). Now by Theorem F we obtain $v \equiv u$ in a connected component W of nonsingular (with respect to both v and u) set, containing q . We claim $W = \Omega \setminus \{p_0\}$. Otherwise there is a point $q' \in (S_{\vec{F}}(v) \cap \bar{W}) \setminus \{p_0\}$ ($(S_{\vec{F}}(u) \cap \bar{W}) \setminus \{p_0\}$, resp.) at which $\nabla u + \vec{F} = \nabla v + \vec{F} = 0$ ($\nabla v + \vec{F} = \nabla u + \vec{F} = 0$, resp.). Therefore $q' \in S_{\vec{F}}(u)$ ($q' \in S_{\vec{F}}(v)$, resp.), a contradiction to p_0 being the only singular point of u (v , resp.). Hence $v \equiv u$ in Ω .

Case 2: Suppose $v > u$ in $\Omega \setminus \{p_0\}$. So there is a small ball B centered at p_0 such that $v > u$ in $B \setminus \{p_0\}$. It follows that

$$v \geq u + c$$

on ∂B for some constant $c > 0$. By Theorem C'' (a version of the usual maximum principle), we conclude that $v \geq u + c$ in B . But $v(p_0) = u(p_0)$ implies $0 \geq c$, a contradiction. We have shown the impossibility of this case. \square

Proof. (of Theorem K) We first observe that the assumption $\mathcal{H}_{m-1}(\overline{S_{\vec{F}}(u)}) = 0$ or $\mathcal{H}_{m-1}(\overline{S_{\vec{F}}(v)}) = 0$ in Theorem J is satisfied for $m = 2n$ and $\vec{F} = (-x^2, x^1, \dots, -x^{2n}, x^{2n-1})$ by Lemma 5.4 in [12]. Also $\text{div } \vec{F}^b = 2n > 0$. In view of Corollary D, Corollary I, and Theorem J, we finally reach a situation that Σ_1 and Σ_2 are tangent at a nonisolated singular point for both Σ_1 and Σ_2 if they don't coincide completely. This contradicts the assumption that *either Σ_1 or Σ_2 has only isolated singular points*. \square

Lemma 5.1. *Suppose $|d\psi|_* \neq 0$, $|d\phi|_* \neq 0$. Then the following formula*

$$(5.1) \quad \begin{aligned} & \langle d\psi - d\phi, \frac{d\psi}{|d\psi|_*} - \frac{d\phi}{|d\phi|_*} \rangle^* \\ &= \frac{1}{2}(|d\psi|_* + |d\phi|_*) \left| \frac{d\psi}{|d\psi|_*} - \frac{d\phi}{|d\phi|_*} \right|_*^2 \end{aligned}$$

holds.

Proof. We compute

$$(5.2) \quad \begin{aligned} & \langle d\psi - d\phi, \frac{d\psi}{|d\psi|_*} - \frac{d\phi}{|d\phi|_*} \rangle^* \\ &= |d\psi|_* + |d\phi|_* - \frac{\langle d\phi, d\psi \rangle^*}{|d\psi|_*} - \frac{\langle d\psi, d\phi \rangle^*}{|d\phi|_*} \\ &= (|d\psi|_* + |d\phi|_*)(1 - \cos \vartheta) \end{aligned}$$

where we write $\langle d\phi, d\psi \rangle^* = \langle d\psi, d\phi \rangle^* = |d\psi|_* |d\phi|_* \cos \vartheta$. On the other hand, we have

$$(5.3) \quad \begin{aligned} & \left| \frac{d\psi}{|d\psi|_*} - \frac{d\phi}{|d\phi|_*} \right|_*^2 \\ &= \left| \frac{d\psi}{|d\psi|_*} \right|_*^2 + \left| \frac{d\phi}{|d\phi|_*} \right|_*^2 - 2 \frac{\langle d\psi, d\phi \rangle^*}{|d\psi|_* |d\phi|_*} \\ &= 2(1 - \cos \vartheta) \end{aligned}$$

Now (5.1) follows from (5.2) and (5.3). □

We remark that the formula (5.1) in vector form first appeared in [26], [23], and [16] independently. The version for Heisenberg group appeared in Lemma 5.1' of [12]). For the case of bounded variation, the reader is referred to [10]. For the definition of $S(u)$ ($S(v)$, resp.), see the paragraph before Theorem J' in Section 1.

Lemma 5.2. *Suppose $(M, \langle \cdot, \cdot \rangle^*, dv_M)$ of dimension $m+1$ has isometric translations Ψ_a near $p_0 \in M$, transversal to a hypersurface Σ passing through p_0 . Take a system of translation-isometric coordinates x^1, x^2, \dots, x^{m+1} in an open neighborhood Ω of p_0 such that $x^{m+1} = 0$ on $\Sigma \cap \Omega$. Let $V \subset \bar{V} \subset \Omega$ be a smaller open neighborhood of p_0 . Let $v, u \in C^2(\Sigma \cap V) \cap C^0(\overline{\Sigma \cap V})$ define graphs in Ω and satisfy*

$$(5.4) \quad H(v) \leq H(u) \text{ in } (\Sigma \cap V) \setminus \{S(u) \cup S(v)\},$$

$$(5.5) \quad v \geq u \text{ on } \partial(\Sigma \cap V).$$

Assume $\mathcal{H}_{m-1}(\overline{S(u) \cup S(v)}) = 0$. Let $\psi := v - x^{m+1}$, $\phi := u - x^{m+1}$. Then $\frac{d\psi}{|d\psi|_} = \frac{d\phi}{|d\phi|_*} \bmod \ker G$ in $(\Sigma \cap V)^+ \setminus \overline{S(u) \cup S(v)}$ where $(\Sigma \cap V)^+ := \{q \in \Sigma \cap V : u(q) - v(q) > 0\}$.*

Proof. First $\mathcal{H}_{m-1}(\overline{S(u) \cup S(v)}) = 0$ means that given any $\varepsilon > 0$, we can find countably many ball $B_{j,\varepsilon}$, $j = 1, 2, \dots$ such that $\overline{S(u) \cup S(v)} \subset \bigcup_{j=1}^{\infty} B_{j,\varepsilon}$ and $\sum_{j=1}^{\infty} \mathcal{H}_{m-1}(\partial B_{j,\varepsilon}) < \varepsilon$ and we can arrange $\bigcup_{j=1}^{\infty} B_{j,\varepsilon_1} \subset \bigcup_{j=1}^{\infty} B_{j,\varepsilon_2}$ for $\varepsilon_1 < \varepsilon_2$. Since $\overline{S(u) \cup S(v)}$ is compact, we can find finitely many $B_{j,\varepsilon}$'s, say $j = 1, 2, \dots, n(\varepsilon)$, still covering $\overline{S(u) \cup S(v)}$. Suppose $(\Sigma \cap V)^+$ is not empty. Then by Sard's theorem there exists a sequence of positive number δ_i converging to 0 as i goes to infinity, such that $(\Sigma \cap V)_i^+ := \{p \in \Sigma \cap V : u(p) - v(p) > \delta_i\}$ is not empty and $\partial(\Sigma \cap V)_i^+ \setminus (S(u) \cup S(v))$ is C^2 smooth. Note that $\partial(\Sigma \cap V)_i^+ \cap \partial(\Sigma \cap V) \subset S(u) \cup S(v)$ by (5.5). Choose $a > 0$ (independent of ε and δ_i) such that $[(\Sigma \cap V)_i^+ \setminus \bigcup_{j=1}^{n(\varepsilon)} B_{j,\varepsilon}] \times [-a, a] \subset \Omega$. Now we consider

$$(5.6) \quad I_\varepsilon^i := \oint_{\partial[(\Sigma \cap V)_i^+ \setminus \bigcup_{j=1}^{n(\varepsilon)} B_{j,\varepsilon}] \times [-a, a]} \tan^{-1}(\psi - \phi) \left(\frac{d\psi}{|d\psi|_*} - \frac{d\phi}{|d\phi|_*} \right) \lrcorner dv_M.$$

By Stokes' theorem we have

$$(5.7) \quad I_\varepsilon^i = \int_{[(\Sigma \cap V)_i^+ \setminus \bigcup_{j=1}^{n(\varepsilon)} B_{j,\varepsilon}] \times [-a, a]} \left\{ \frac{d\psi - d\phi}{1 + (\psi - \phi)^2} \wedge \left[\left(\frac{d\psi}{|d\psi|_*} - \frac{d\phi}{|d\phi|_*} \right) \lrcorner dv_M \right] + \tan^{-1}(\psi - \phi) (H_\psi - H_\phi) dv_M \right\}$$

Observe that $\psi - \phi = v - u < -\delta_i < 0$ in $(\Sigma \cap V)_i^+ \times [-a, a]$ and $H_\psi(q) - H_\phi(q) = H(v)(\pi_\Sigma(q)) - H(u)(\pi_\Sigma(q)) \leq 0$ by Corollary 3.2 and (5.4), where π_Σ is the projection to Σ $((x^1, \dots, x^m, x^{m+1}) \rightarrow (x^1, \dots, x^m))$. So the second term in the right hand side of (5.7) is nonnegative. As to the first term, we observe that $\eta \wedge (\omega \lrcorner dv_M) = \langle \eta, \omega \rangle^* dv_M$ for any 1-forms η and ω . It then follows from (5.1) in Lemma 5.1 and (5.7) that

$$(5.8) \quad I_\varepsilon^i \geq \int_{[(\Sigma \cap V)_i^+ \setminus \bigcup_{j=1}^{n(\varepsilon)} B_{j,\varepsilon}] \times [-a, a]} \frac{|d\psi|_* + |d\phi|_*}{2[1 + (\psi - \phi)^2]} \left| \frac{d\psi}{|d\psi|_*} - \frac{d\phi}{|d\phi|_*} \right|_*^2 dv_M$$

On the other hand, look at the boundary integral in (5.6): $\psi - \phi = v - u = \delta_i$ on $\partial(\Sigma \cap V)_i^+ \times [-a_i, a_i] \rightarrow 0$ as $i \rightarrow \infty$, the boundary integral on $[(\Sigma \cap V)_i^+ \setminus \bigcup_{j=1}^{n(\varepsilon)} B_{j,\varepsilon}] \times \{-a_i\}$ cancels with the boundary integral on $[(\Sigma \cap V)_i^+ \setminus \bigcup_{j=1}^{n(\varepsilon)} B_{j,\varepsilon}] \times \{a_i\}$ due to translation invariance and orientation, and $\frac{d\psi}{|d\psi|_*} - \frac{d\phi}{|d\phi|_*}$ is bounded while $\sum_{j=1}^{n(\varepsilon)} \mathcal{H}_{m-1}(\partial B_{j,\varepsilon}) < \varepsilon$. So we conclude that $I_\varepsilon^i \leq \varepsilon$ for $i = i(\varepsilon)$ large enough from (5.6). Together with (5.8) we conclude that

$$\left| \frac{d\psi}{|d\psi|_*} - \frac{d\phi}{|d\phi|_*} \right|_* = 0$$

It follows that $\frac{d\psi}{|d\psi|_*} = \frac{d\phi}{|d\phi|_*} \pmod{\ker G}$ in $(\Sigma \cap V)^+ \setminus \overline{S(u) \cup S(v)}$.

□

Lemma 5.3. *Suppose $(M, \langle \cdot, \cdot \rangle^*, dv_M)$ of dimension $m + 1$ has isometric translations Ψ_a near $p_0 \in M$, transversal to a hypersurface Σ passing through p_0 . Take a system of translation-isometric coordinates x^1, x^2, \dots, x^{m+1} in an open neighborhood Ω of p_0 such that $x^{m+1} = 0$ on $\Sigma \cap \Omega$ and p_0 is the origin $(0, \dots, 0)$. Let $v, u : \Sigma \cap \Omega \rightarrow \mathbb{R}$ be two graphs in Ω . Let $\psi := v(x^1, x^2, \dots, x^m) - x^{m+1}$, $\phi :=$*

$u(x^1, x^2, \dots, x^m) - x^{m+1}$. Suppose $\frac{d\psi}{|d\psi|_*} = \frac{d\phi}{|d\phi|_*} \mod \ker G$. Moreover, we assume the rank condition (1.7). Then $\nabla v = \nabla u$ in $\Sigma \cap \Omega$.

Proof. Note that $\xi = \cap_{\theta \in \ker G} \ker \theta$. So $\frac{d\psi}{|d\psi|_*} = \frac{d\phi}{|d\phi|_*} \mod \ker G$ implies $\xi \cap T_p\{\psi = c(p)\} = \xi \cap T_p\{\phi = 0\}$. By a translation (depending on p) in the x^{m+1} direction, we can translate the hypersurface $\{\psi = c(p)\}$ to the fixed hypersurface $\{\psi = 0\}$. Since translations are isometries, we then have

$$(5.9) \quad \xi \cap T_{(\bar{p}, v(\bar{p}))}\{\psi = 0\} = \Phi(\bar{p})_*(\xi \cap T_{(\bar{p}, u(\bar{p}))}\{\phi = 0\})$$

where $\bar{p} \in \Sigma \cap \Omega$ and $\Phi(\bar{p})$ is a translation in the x^{m+1} direction, depending on \bar{p} . By (1.7) we have $\mathcal{L}(\xi \cap T_{(\bar{p}, v(\bar{p}))}\{\psi = 0\}) = \Gamma(T_{(\bar{p}, v(\bar{p}))}\{\psi = 0\}) (\mathcal{L}(\xi \cap T_{(\bar{p}, u(\bar{p}))}\{\phi = 0\}) = \Gamma(T_{(\bar{p}, u(\bar{p}))}\{\phi = 0\}))$, resp.) since $\text{rank}(\mathcal{L}(\xi \cap T_{(\bar{p}, v(\bar{p}))}\{\psi = 0\})) = m$ ($\text{rank}(\mathcal{L}(\xi \cap T_{(\bar{p}, u(\bar{p}))}\{\phi = 0\})) = m$, resp.) and $\xi \cap T_{(\bar{p}, v(\bar{p}))}\{\psi = 0\} \subset T_{(\bar{p}, v(\bar{p}))}\{\psi = 0\}$ ($\xi \cap T_{(\bar{p}, u(\bar{p}))}\{\phi = 0\} \subset T_{(\bar{p}, u(\bar{p}))}\{\phi = 0\}$, resp.). Here $\Gamma(E)$ denotes the space of all (C^∞ smooth) sections of the vector bundle E . It follows from (5.9) that

$$T_{(\bar{p}, v(\bar{p}))}\{\psi = 0\} = \Phi(\bar{p})_*(T_{(\bar{p}, u(\bar{p}))}\{\phi = 0\}).$$

We then have $\nabla v = \nabla u$ in $\Sigma \cap \Omega$. □

Proof. (of Theorem \tilde{C}'') By Lemma 5.2 we get $\frac{d\psi}{|d\psi|_*} = \frac{d\phi}{|d\phi|_*} \mod \ker G$ in $(\Sigma \cap V)^+ \setminus \overline{S(u) \cup S(v)}$ where $(\Sigma \cap V)^+ := \{q \in \Sigma \cap V : u(q) - v(q) > 0\}$. By Lemma 5.3 we get $\nabla v = \nabla u$ in $(\Sigma \cap V)^+$ and hence $u - v = \text{constant} > 0$ in $(\Sigma \cap V)^+$. On the other hand, we have $v \geq u$ on $\partial(\Sigma \cap V)$ by assumption. We get contradiction by continuity of v and u . So $(\Sigma \cap V)^+$ is an empty set. We then conclude that $v \geq u$ in $\Sigma \cap V$. □

Proof. (of Theorem J') The idea is similar as in the proof of Theorem J.

Case 1: Suppose $v = u$ at $q \neq p_0$. Observe that q is nonsingular with respect to u (v , resp.) since p_0 is the only singular point of u (v , resp.). On the other hand, we have $\nabla(v - u) = 0$ at q since $v - u \geq 0$ and $v - u = 0$ at q . It follows that

$$\begin{aligned} d\psi &= dv - dx^{m+1} \\ &= du - dx^{m+1} = d\phi \end{aligned}$$

at $(q, v(q)) = (q, u(q))$. So $\xi \not\subset \ker d\phi$ at $(q, u(q))$ implies $\xi \not\subset \ker d\psi$ at $(q, v(q))$. That is, q is also nonsingular with respect to v (u , resp.). By Theorem C' we obtain $v \equiv u$ in a connected component W of nonsingular (with respect to both v and u) set, containing q . We claim $W = (\Sigma \cap \Omega) \setminus \{p_0\}$. Otherwise there is a point $q' \in (S_{\Sigma \cap \Omega}(v) \cap \bar{W}) \setminus \{p_0\}$ ($(S_{\Sigma \cap \Omega}(u) \cap \bar{W}) \setminus \{p_0\}$, resp.) at which $\nabla v = \nabla u$, and hence $\xi \subset \ker d\psi = \ker d\phi$ at $(q', v(q')) = (q', u(q'))$. So q' is also a singular point of u (v , resp.), a contradiction to p_0 being the only singular point of u (v , resp.). Hence $v \equiv u$ in $\Sigma \cap \Omega$.

Case 2: Suppose $v > u$ in $(\Sigma \cap \Omega) \setminus \{p_0\}$. So there is a small ball $B \subset \Sigma \cap \Omega$, centered at p_0 such that $v > u$ in $B \setminus \{p_0\}$. It follows that

$$v \geq u + c$$

on ∂B for some constant $c > 0$. By Theorem \tilde{C}'' , we conclude that $v \geq u + c$ in B . But $v(p_0) = u(p_0)$ implies $0 \geq c$, a contradiction. We have shown the impossibility of this case. \square

6. APPLICATIONS: UNIQUENESS AND NONEXISTENCE

Consider the Heisenberg cylinder $H_n \setminus \{0\}$ with CR structure same as H_n and contact form

$$\theta = \frac{1}{\rho^2} \Theta,$$

denoted as $(H_n \setminus \{0\}, \rho^{-2} \Theta)$, where $\Theta := dz + \sum_{j=1}^n (x_j dy_j - y_j dx_j)$ and $\rho := [(\sum_{j=1}^n (x_j^2 + y_j^2))^2 + 4z^2]^{1/4}$. Here $x_1, y_1, \dots, x_n, y_n, z$ denote the coordinates of H_n . Topologically $H_n \setminus \{0\}$ is homeomorphic to $S^{2n} \times R^+$ through the map

$$(x_1, y_1, \dots, x_n, y_n, z) \rightarrow \left(\left(\frac{x_1}{\rho}, \dots, \frac{y_n}{\rho}, \frac{z}{\rho^2} \right), \rho \right)$$

where the Heisenberg sphere $S^{2n} \subset H_n$ is defined by $\rho = 1$. Next we want to compute horizontal (p -) mean curvature of a hypersurface of $(H_n \setminus \{0\}, \rho^{-2} \Theta)$, described by a defining function ϕ . Take an orthonormal basis $e_I := \rho \hat{e}_I$, $1 \leq I \leq 2n$, with respect to the Levi metric (see Subsection B in the Appendix) and $T := \rho^2 \frac{\partial}{\partial z}$ where

$$\begin{aligned} \hat{e}_I &= \hat{e}_j = \frac{\partial}{\partial x_j} + y_j \frac{\partial}{\partial z}, \text{ for } 1 \leq I = j \leq n \\ \hat{e}_I &= \hat{e}_{j'} = \frac{\partial}{\partial y_j} - x_j \frac{\partial}{\partial z}, \text{ for } n+1 \leq I = j' = n+j \leq 2n. \end{aligned}$$

So the dual coframe is $\theta^I = \rho^{-1} dx^I$ ($x^j = x_j$, $x^{n+j} = y_j$ for $1 \leq j \leq n$) and $\theta = \rho^{-2} \Theta$ and the associated subriemannian metric $\langle \cdot, \cdot \rangle^*$ on $H_n \setminus \{0\}$ is given by

$$\langle \theta^I, \theta^K \rangle^* = \delta_{IK}, \quad \langle \cdot, \theta \rangle^* = \langle \theta, \cdot \rangle^* = 0$$

(cf. (7.6)). We have the volume form

$$(6.1) \quad dV := \theta^1 \wedge \theta^2 \wedge \dots \wedge \theta^{2n} \wedge \theta = \rho^{-(2n+2)} dx^1 \wedge dx^2 \wedge \dots \wedge dx^{2n} \wedge \Theta.$$

Compute

$$\begin{aligned} d\phi &= (e_I \phi) \theta^I + (T\phi) \theta \\ &= (\hat{e}_I \phi) dx^I + \frac{\partial \phi}{\partial z} \Theta, \end{aligned} \quad (6.2)$$

and hence

$$\begin{aligned} |d\phi|_*^2 &: = \langle d\phi, d\phi \rangle^* \\ &= \sum_{I=1}^{2n} (e_I \phi)^2 = \rho^2 \sum_{I=1}^{2n} (\hat{e}_I \phi)^2. \end{aligned} \quad (6.3)$$

From (6.1), 6.2, we compute

$$\begin{aligned}
(6.4) \quad & \frac{d\phi}{|d\phi|_*} \lrcorner dV \\
&= (-1)^{I-1} \frac{e_I \phi}{|d\phi|_*} \theta^1 \wedge \dots \wedge \hat{\theta}^I \wedge \dots \wedge \theta^{2n} \wedge \theta \\
&\quad (\hat{\theta}^I \text{ means } \theta^I \text{ deleted}) \\
&= (-1)^{I-1} \frac{\hat{e}_I \phi}{|d\phi|_{H_n}} \rho^{-(2n+1)} dx^1 \wedge \dots \wedge d\hat{x}^I \wedge \dots \wedge dx^{2n} \wedge \Theta \\
&= \rho^{-(2n+1)} \frac{d\phi}{|d\phi|_{H_n}} \lrcorner_{H_n} dV_{H_n}.
\end{aligned}$$

where dV_{H_n} denotes the standard volume form of H_n , which is $dx^1 \wedge \dots \wedge d\hat{x}^I \wedge \dots \wedge dx^{2n} \wedge \Theta$ and $|\cdot|_{H_n}$ denotes the length with respect to the standard subriemannian metric in H_n (see (7.5)). It follows that $|d\phi|_{H_n} = (\sum_{I=1}^{2n} (\hat{e}_I \phi)^2)^{1/2}$. Taking exterior differentiation of (6.4) gives

$$\begin{aligned}
(6.5) \quad H_\phi dV &: = d\left(\frac{d\phi}{|d\phi|_*} \lrcorner dV\right) \\
&= \rho^{-(2n+1)} d\left(\frac{d\phi}{|d\phi|_{H_n}} \lrcorner_{H_n} dV_{H_n}\right) \\
&\quad + \frac{\hat{e}_I \phi}{|d\phi|_{H_n}} \hat{e}_I (\rho^{-(2n+1)}) dV_{H_n} \\
&= [\rho^{-(2n+1)} \hat{H}_\phi - (2n+1) \rho^{-2n-2} \frac{\hat{e}_I \phi}{|d\phi|_{H_n}} \hat{e}_I \rho] dV_{H_n}
\end{aligned}$$

where \hat{H}_ϕ denotes the horizontal (p -) mean curvature with respect to the standard subriemannian metric in H_n . Observe that $\frac{\hat{e}_I \phi}{|d\phi|_{H_n}} \hat{e}_I$ is the horizontal normal to hypersurfaces defined by $\phi = \text{constant}$, denoted as e_{2n}^ϕ . By (6.1) and (6.5), we obtain

$$(6.6) \quad H_\phi = \rho \hat{H}_\phi - (2n+1) e_{2n}^\phi \rho$$

(cf. Lemma 7.2 in [12]). For $\phi = u(r) - z$ where $r = [\sum_{I=1}^{2n} (x^I)^2]^{1/2}$, we want to get a formula for H_ϕ in terms of u and its derivatives. First we compute $|d\phi|_{H_n}$ as follows:

$$\begin{aligned}
(6.7) \quad |d\phi|_{H_n}^2 &= \sum_{I=1}^{2n} (\hat{e}_I \phi)^2 \\
&= \sum_{I=1}^{2n} (u'(r) \partial_I r - x^{I'})^2 \\
&= \sum_{I=1}^{2n} (u'(r) \frac{x^I}{r} - x^{I'})^2 = (u'(r))^2 + r^2
\end{aligned}$$

where $x^{I'} := x^{n+j}$ for $I = j$, $x^{I'} := -x^j$ for $I = n+j$, $1 \leq j \leq n$ and note that $\sum_{I=1}^{2n} x^I x^{I'} = 0$. Next noting that $\partial_z(\frac{\hat{e}_I \phi}{|d\phi|_{H_n}}) = 0$, we compute

$$\begin{aligned} (6.8) \quad \hat{H}_\phi &= \hat{e}_I \left(\frac{\hat{e}_I(\phi)}{|d\phi|_{H_n}} \right) \\ &= \operatorname{div}_{R^{2n}} \left(\frac{u'(r) \nabla r - (x^{I'})}{\sqrt{(u'(r))^2 + r^2}} \right) \end{aligned}$$

by (6.7). Observe that $\partial_I x^{I'} = 0$ and

$$\begin{aligned} &\sum_{I=1}^{2n} x^{I'} \partial_r (\sqrt{(u'(r))^2 + r^2})^{-1} \partial_I r \\ &= r^{-1} \partial_r (\sqrt{(u'(r))^2 + r^2})^{-1} \sum_{I=1}^{2n} x^{I'} x^I = 0. \end{aligned}$$

So we can reduce (6.8) to

$$\begin{aligned} (6.9) \quad \hat{H}_\phi &= \operatorname{div}_{R^{2n}} \left(\frac{u'(r) \nabla r}{\sqrt{(u'(r))^2 + r^2}} \right) \\ &= \operatorname{div}_{R^{2n}} \left(\frac{u'(r) r^{2n-1}}{\sqrt{(u'(r))^2 + r^2}} \frac{\nabla r}{r^{2n-1}} \right) \\ &= \left(\nabla \frac{u'(r) r^{2n-1}}{\sqrt{(u'(r))^2 + r^2}} \right) \cdot \frac{\nabla r}{r^{2n-1}} \quad (\text{since } \operatorname{div}_{R^{2n}} \left(\frac{\nabla r}{r^{2n-1}} \right) = 0) \\ &= \frac{d}{dr} \left(\frac{u'(r) r^{2n-1}}{\sqrt{(u'(r))^2 + r^2}} \right) \frac{1}{r^{2n-1}} \quad (\text{since } |\nabla r|^2 = 1). \end{aligned}$$

On the other hand, we compute

$$(6.10) \quad e_{2n}^\phi \rho = \frac{\hat{e}_I \phi}{|d\phi|_{H_n}} \hat{e}_I \rho = \frac{r^2 (ru'(r) - 2u(r))}{\rho^3 \sqrt{(u'(r))^2 + r^2}}$$

for $\phi = u(r) - z$. Substituting (6.9) and (6.10) into (6.6), we obtain

$$\begin{aligned} (6.11) \quad H_\phi &= \frac{\rho}{r^{2n-1}} \frac{d}{dr} \left(\frac{u'(r) r^{2n-1}}{\sqrt{(u'(r))^2 + r^2}} \right) \\ &\quad - (2n+1) \frac{r^2 (ru'(r) - 2u(r))}{\rho^3 \sqrt{(u'(r))^2 + r^2}}. \end{aligned}$$

For $\phi = u(r) - z$ with $u(r) = cr^2$, c being a constant, we get

$$\begin{aligned} (6.12) \quad H_\phi &= \frac{2(2n-1)c \rho}{\sqrt{1+4c^2} r} \\ &= \frac{2(2n-1)c}{(1+4c^2)^{1/4}} \end{aligned}$$

at points where $\phi = cr^2 - z = 0$. That is to say, the hypersurface in the Heisenberg cylinder $(H_n \setminus \{0\}, \rho^{-2}\Theta)$, defined by $z = cr^2$, has constant horizontal (p -) mean curvature as shown in (6.12) (note that at $p_0 \in \Sigma := \{\phi = c\}$, we have $H_\phi(p_0) =$

$H_\Sigma(p_0)$, the horizontal (p -) mean curvature of Σ . See Proposition B.1 in the Appendix). Next we want to compute the horizontal (p -) mean curvature of Heisenberg spheres defined by $\rho^4 = c > 0$. Let $\phi = \rho^4 - c$. We compute

$$\begin{aligned}\hat{e}_I \phi &= (\partial_I + x^{I'} \partial_z)((r^2)^2 + 4z^2) \\ &= 4(r^2 x^I + 2x^{I'} z).\end{aligned}$$

It follows that

$$\begin{aligned}(6.13) \quad |d\phi|_{H_n}^2 &= \sum_{I=1}^{2n} (\hat{e}_I \phi)^2 \\ &= 16(r^6 + 4r^2 z^2) = 16r^2 \rho^4.\end{aligned}$$

Then a straightforward computation shows

$$(6.14) \quad \hat{H}_\phi = (2n+1) \frac{r}{\rho^2}.$$

On the other hand, since $4\rho^3 \hat{e}_I \rho = \hat{e}_I \phi$ for $\phi = \rho^4 - c$, we have

$$\begin{aligned}(6.15) \quad e_{2n}^\phi \rho &= \sum_{I=1}^{2n} \frac{\hat{e}_I \phi}{|d\phi|_{H_n}} \hat{e}_I \rho \\ &= \sum_{I=1}^{2n} \frac{(\hat{e}_I \phi)^2}{|d\phi|_{H_n} (4\rho^3)} \\ &= \frac{|d\phi|_{H_n}}{4\rho^3} = \frac{r}{\rho}\end{aligned}$$

by (6.13). Substituting (6.14) and (6.15) into (6.6) gives

$$H_\phi = \rho(2n+1) \frac{r}{\rho^2} - (2n+1) \frac{r}{\rho} = 0.$$

So this means that Heisenberg spheres $\{\rho^4 = c\}$ are horizontally (p -) minimal hypersurfaces in the Heisenberg cylinder $(H_n \setminus \{0\}, \rho^{-2}\Theta)$. We summarize what we obtain so far as follows:

Proposition 6.1. *Let Σ be a hypersurface in the Heisenberg cylinder $(H_n \setminus \{0\}, \rho^{-2}\Theta)$ with $n \geq 1$. We have*

- (a) *Suppose Σ is defined by $z = cr^2$ for a constant c . Then Σ is a hypersurface of constant horizontal (p -) mean curvature with constant $\frac{2(2n-1)c}{(1+4c^2)^{1/4}}$;*
- (b) *Suppose Σ is defined by $\rho^4 = c$ for a constant $c > 0$. Then Σ is a horizontally (p -) minimal hypersurface.*

Observe that the dilation $\tau_\lambda : (x^1, \dots, x^{2n}, z) \rightarrow (\lambda x^1, \dots, \lambda x^{2n}, \lambda^2 z)$ preserves $\rho^{-2}\Theta$, i.e., $\tau_\lambda^*(\rho^{-2}\Theta) = \rho^{-2}\Theta$ for any $\lambda \in \mathbb{R} \setminus \{0\}$. So τ_λ is a pseudohermitian isomorphism of the Heisenberg cylinder $(H_n \setminus \{0\}, \rho^{-2}\Theta)$. We can now prove a uniqueness result stated in Theorem L in Section 1.

Proof. (of Theorem L) For the case (a), we take a Heisenberg sphere $S(c_1)$ defined by $\rho^4 = c_1$ for c_1 large enough so that the interior region $\{\rho^4 < c_1\}$ of $S(c_1)$ contains

Σ . Decrease (or take) c_1 to reach a constant $c_2 > 0$ so that $S(c_2)$ is tangent to Σ at some point p_0 while Σ lies in $\{\rho^4 \leq c_2\}$. Observe that

$$H_\Sigma \leq 0 = H_{S(c_2)}$$

near p_0 by the assumption and Proposition 6.1 (b). It follows from the SMP (Theorem C and Theorem J') that Σ must coincide with $S(c_2)$.

Similarly for the case (b), we can first find a Heisenberg sphere $S(c_3)$ defined by $\rho^4 = c_3$ for c_3 small enough so that $S(c_3)$ is contained in the interior region of Σ . Increase (or take) c_3 to reach a constant $c_4 > 0$ so that $S(c_4)$ is tangent to Σ at some point q while $\{\rho^4 \leq c_4\}$ is contained in the interior region of Σ . Now observe that

$$H_\Sigma \geq 0 = H_{S(c_4)}$$

near q by the assumption and Proposition 6.1 (b). So it follows that $\Sigma = S(c_4)$ by the SMP. \square

We can also show a nonexistence result (pseudo-halfspace theorem).

Proof. (of Theorem N) For the first case $\Omega = \{z > \varphi(\sqrt{x_1^2 + \dots + x_{2n}^2})\}$, we consider comparison hypersurfaces -horizontal hyperplanes $\{z = c, \text{ a constant}\}$. Starting from $c = c_0 < \min_{\tau \in [0, \infty)} \varphi(\tau)$ (existence by the assumption: $\lim_{\tau \rightarrow \infty} \varphi(\tau) = \infty$), we increase c to reach $c = c_1$ such that the hyperplane $\{z = c_1\}$ is tangent to Σ at some point p_1 at the first time. The existence of such $p_1 \in \{z = c_1\} \cap \Sigma$ is due to the immersion being proper. Note that the hyperplane $\{z = c_1\}$ is horizontally (p -) minimal and its singular set consists of one isolated singular point $(0, \dots, 0, c_1)$. We can then apply the SMP (Corollary I or Theorem J) at p_1 to conclude that $\Sigma \subset$ the hyperplane $\{z = c_1\}$ which touches $\partial\Omega = \{z = \varphi(\sqrt{x_1^2 + \dots + x_{2n}^2})\}$. But such Σ is not properly immersed in Ω .

For the second case $\Omega = \{x_1 > \varphi(\sqrt{x_2^2 + \dots + x_{2n}^2 + z^2})\}$, we consider comparison hypersurfaces -vertical hyperplanes $\{x_1 = c, \text{ a constant}\}$. By a similar reasoning as for the first case, we can find $c = c_1$ such that the hyperplane $\{x_1 = c_1\}$ is tangent to Σ at some point q . Note that the hyperplane $\{x_1 = c_1\}$ is horizontally (p -) minimal and has no singular points. Apply the SMP (Corollary D) to this situation to conclude that $\Sigma \subset \{x_1 = c_1\}$ which touches $\partial\Omega = \{x_1 = \varphi(\sqrt{x_2^2 + \dots + x_{2n}^2 + z^2})\}$. But such Σ is not properly immersed in Ω . \square

The simplest example is $\varphi(\tau) = a\tau$ with $a > 0$. Call associated domains wedge-shaped. Theorem N tells us nonexistence of p -minimal hypersurfaces in wedge-shaped domains. But Theorem N does not hold for the case $a = 0$. That is, halfspace theorem does not hold since there are catenoid type horizontal (p -) minimal hypersurfaces with finite height ([34]) in H_n for $n \geq 2$. On the other hand, we do have halfspace theorem for H_1 (see [9]).

7. APPENDIX

A: Bony's strong maximum principle

For completeness and reader's convenience, we review some material in Bony's original paper [5].

Let Ω be a domain of R^n . We consider a differential operator of second order:

$$Lu(x) = \sum_{i,j=1}^n a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j}(x) + \sum_{i=1}^n a_i(x) \frac{\partial u}{\partial x_i}(x) + a(x)u(x)$$

with the following properties:

(α) The quadratic form $(a_{ij}(x))$ is nonnegative for each $x \in \Omega$, that is,

$$\sum_{i,j=1}^n a_{ij}(x) \xi_i \xi_j \geq 0 \quad \text{for any } x \in \Omega, \xi \in R^n.$$

(β) $a(x) \leq 0$ in Ω and $a(x) \in C^\infty$.

(γ) There exist vector fields X_1, \dots, X_r and Y of class C^∞ such that

$$Lu = \sum_{k=1}^r X_k^2 u + Y u + a u.$$

We remark that an operator L with property (α) may not have property (γ). Writing $X_k = b_k^j \frac{\partial}{\partial x_j}$ (summation convention), we compute

$$\sum_{k=1}^r X_k^2 u = \sum_{k=1}^r b_k^j b_k^i \frac{\partial^2 u}{\partial x_i \partial x_j} + b_k^j \frac{\partial b_k^i}{\partial x_j} \frac{\partial u}{\partial x_i}.$$

Therefore we have

$$(7.1) \quad a_{ij} = \sum_{k=1}^r b_k^j b_k^i.$$

Lemma A1. $\sum_{i,j=1}^n a_{ij}(x) \xi_i \xi_j = 0$ if and only if $\xi = (\xi_i)$ is perpendicular to X_k for any k , $1 \leq k \leq r$.

Proof. By (7.1) we have

$$(7.2) \quad \begin{aligned} \sum_{i,j=1}^n a_{ij}(x) \xi_i \xi_j &= \sum_{k=1}^r \sum_{i,j=1}^n b_k^j b_k^i \xi_i \xi_j \\ &= \sum_{k=1}^r \left(\sum_{j=1}^n b_k^j \xi_j \right) \left(\sum_{i=1}^n b_k^i \xi_i \right) \\ &= \sum_{k=1}^r (X_k \cdot \xi)^2 \end{aligned}$$

where " \cdot " means the standard inner product on R^n . The result follows from (7.2). \square

From (7.2) we also learn that (γ) implies (α).

Proposition A2 (Proposition 1.1 in [5]). *Suppose $u \in C^2(\Omega)$ attains a nonnegative local maximum at $x_0 \in \Omega$. Then we have $Lu(x_0) \leq 0$. If we further assume the maximum is positive and $a(x_0) < 0$. Then we have $Lu(x_0) < 0$.*

Proof. Observe that the matrix $(u_{ij}(x_0))$ is nonpositive and the matrix $(a_{ij}(x_0))$ is nonnegative. It follows that

$$\sum_{i,j=1}^n a_{ij}(x_0)u_{ij}(x_0) = \text{Trace}[(a_{ij}(x_0))(u_{ij}(x_0))] \leq 0$$

by elementary linear algebra. We then have

$$\begin{aligned} Lu(x_0) &= \sum_{i,j=1}^n a_{ij}(x_0)u_{ij}(x_0) + \sum_{i=1}^n a_i(x_0)u_i(x_0) + a(x_0)u(x_0) \\ &\leq 0 \end{aligned}$$

in which we have used $u_i(x_0) = 0$.

□

Let F be a closed subset of Ω . We say a vector ν is normal to F at a point $x_0 \in F$ if there exists a ball $B \subset \Omega \setminus F$ with center x_1 , such that $x_0 \in \partial B$ and the vectors $x_1 - x_0$ and ν are parallel.

We say a vector field X is tangent to F if for any $x_0 \in F$ and any vector ν normal to F at x_0 , $X(x_0)$ is perpendicular to ν .

Theorem A3 (Theorem 2.1 in [5]) *Let Ω be a domain of R^n . Let F be a closed subset of Ω . Suppose X is a Lipschitz continuous vector field on Ω , which is tangent to F . Then any integral curve of X meeting F at a point is entirely contained in F .*

Proof. (outline) Assume, on the contrary, there exists a curve $x(t)$ satisfying

$$\frac{dx(t)}{dt} = X(x(t)),$$

and meeting F at a point, which is not contained in F . Then we can find an interval $[t_0, t_1]$ such that $x(t_0) \in F$ and $x(t) \notin F$ for all $t \in (t_0, t_1]$. We then prove the following two facts:

(a) (Lemma 2.1 in [5]) Let $\delta(t)$ denote the distance between $x(t)$ and F . Then there exists a constant $K_0 > 0$ such that for all $t \in (t_0, t_1]$, there holds

$$\liminf_{h \rightarrow 0} \frac{\delta(t+h) - \delta(t)}{|h|} \geq -K_0\delta(t).$$

(b) (Lemma 2.2 in [5]) Suppose $\varphi(t)$ is a continuous function on $[t_0, t_1]$ and satisfies

$$\liminf_{h \rightarrow 0} \frac{\varphi(t+h) - \varphi(t)}{|h|} \geq -M$$

for all $t \in (t_0, t_1)$. Then φ is Lipschitzian of ratio M on $[t_0, t_1]$.

Now let $\theta = \inf\{t_1 - t_0, \frac{1}{2K_0}\}$ and $\varepsilon = \sup_{t \in [t_0, t_0 + \theta]} \delta(t)$. From (a), (b) we obtain that the function δ is Lipschitzian of ratio $K_0\varepsilon$ on $[t_0, t_0 + \theta]$. So we have

$$(7.3) \quad \begin{aligned} |\delta(t) - \delta(t_0)| &\leq \varepsilon K_0(t - t_0) \\ &\leq \varepsilon K_0\theta \leq \frac{\varepsilon}{2} \end{aligned}$$

for any $t \in [t_0, t_0 + \theta]$. On the other hand, we observe that $\delta(t_0) = 0$ in (7.3) and hence $\varepsilon = \sup_{t \in [t_0, t_0 + \theta]} \delta(t) = \sup_{t \in [t_0, t_0 + \theta]} |\delta(t) - \delta(t_0)| \leq \frac{\varepsilon}{2}$, a contradiction. \square

Let us denote by $\mathcal{L}(X_1, \dots, X_r)$ the smallest C^∞ -module which contains X_1, \dots, X_r and is closed under the Lie bracket. That is, if $Z \in \mathcal{L}(X_1, \dots, X_r)$, then Z is a sum of finite terms of the form

$$(7.4) \quad \lambda[X_{i_1}, [X_{i_2}, \dots, [X_{i_{l-1}}, X_{i_l}]]]$$

where $\lambda \in C^\infty(\Omega)$ and $i_k \in \{1, \dots, r\}$. The rank of $\mathcal{L}(X_1, \dots, X_r)$ at a point x is the dimension of the vector space spanned by the vectors $Z(x)$ for all $Z \in \mathcal{L}(X_1, \dots, X_r)$.

Proposition A4 (Proposition 2.1 in [5]). *Let X_1, \dots, X_r be vector fields of class C^∞ . Take $Z \in \mathcal{L}(X_1, \dots, X_r)$. Then any integral curve of Z can be uniformly approximated by piecewise differentiable curves each of which is an integral curve of one of vector fields X_1, \dots, X_r .*

Theorem A5 (Theorem 2.2 in [5]). *Let Ω be an open set in R^n . Let F be a closed subset of Ω . Suppose X_1, \dots, X_r are vector fields of class C^∞ , each of which is tangent to F . Then for each $Z \in \mathcal{L}(X_1, \dots, X_r)$, Z is tangent to F and any integral curve of Z meeting F at a point is entirely contained in F .*

We refer the reader to the original paper of Bony for the proof of the above two results. We then discuss the propagation of maximums.

Proposition A6 (Proposition 3.1 in [5]) *Let u be a function of class C^2 on Ω such that $Lu \geq 0$. Suppose the maximum of u is nonnegative and attained at a point in Ω . Let F be the set of all points where u attains the maximum. Then for each $k = 1, \dots, r$, the vector field X_k is tangent to F .*

Proof. Let $B(x_0, \rho) \subset \Omega \setminus F$ be a ball of radius ρ and center x_0 , such that $\partial B(x_0, \rho) \cap F = \{x_1\}$. We are going to show

$$\alpha := \sum_{i,j} a_{ij}(x_1)(x_0^i - x_1^i)(x_0^j - x_1^j) = 0.$$

The conclusion follows from Lemma A1 and the definition of tangent vector. Suppose, on the contrary, $\alpha > 0$. Consider an auxiliary function v defined by

$$v(x) = e^{-k|x-x_0|^2} - e^{-k\rho^2}$$

where k is a positive constant. A direct computation gives

$$\begin{aligned} Lv(x_1) &= e^{-k\rho^2} [4k^2 \sum_{i,j} a_{ij}(x_1)(x_0^i - x_1^i)(x_0^j - x_1^j) \\ &\quad - 2k(a_{ii}(x_1) + a_i(x_1)(x_1^i - x_0^i))] \\ &= e^{-k\rho^2} [4k^2\alpha - 2k(a_{ii}(x_1) + a_i(x_1)(x_1^i - x_0^i))] \end{aligned}$$

in which we have used $v(x_1) = 0$. Therefore if k is large enough, we have

$$Lv(x_1) > 0,$$

and hence $Lv > 0$ in a neighborhood V of x_1 . We now consider the function

$$w(x) = u(x) + \lambda v(x), \quad \lambda > 0.$$

It follows that $Lw = Lu + \lambda Lv > 0$ in V since $Lu \geq 0$ by assumption. From Proposition A2 we conclude that w cannot achieve a nonnegative local maximum in V . On the other hand, we are going to show $w|_{\partial V} \leq w(x_1)$, a contradiction.

Let $B(x_0, \rho)^c$ denote the complement of $B(x_0, \rho)$. Observe that for $x \in \partial V \cap B(x_0, \rho)^c$, $v(x) \leq 0$ and hence

$$w(x) \leq u(x) \leq u(x_1) = w(x_1)$$

since u attains the maximum at x_1 and $v(x_1) = 0$. Now observe that $\sup_{\partial V \cap B(x_0, \rho)} u < u(x_1)$ by compactness. So for $x \in \partial V \cap B(x_0, \rho)$, we have

$$w(x) \leq u(x_1) = w(x_1)$$

for small λ .

□

Theorem A7 (Theorem 3.1 in [5]). *Let u be a function of class C^2 in Ω and $Z \in \mathcal{L}(X_1, \dots, X_r)$. Suppose $Lu \geq 0$ and u attains a nonnegative maximum at a point of an integral curve Γ of Z . Then the maximum is attained at all points of Γ .*

Corollary A8 (Corollary 3.1 in [5]) *Suppose the rank of $\mathcal{L}(X_1, \dots, X_r)$ is n for all points. Then a function of class C^2 in Ω satisfying $Lu \geq 0$ cannot achieve a nonnegative maximum in Ω unless it is constant.*

B: Subriemannian geometry from the viewpoint of differential forms

A subriemannian manifold is a (C^∞) smooth manifold M equipped with a non-negative definite inner product $\langle \cdot, \cdot \rangle^*$ on T^*M , its cotangent bundle. Clearly if $\langle \cdot, \cdot \rangle^*$ is positive definite, $(M, \langle \cdot, \cdot \rangle^*)$ is a Riemannian manifold. For M being the Heisenberg group H_n of dimension $m = 2n + 1$, we recall that the multiplication \circ of H_n reads

$$\begin{aligned} (a_1, \dots, a_n, b_1, \dots, b_n, c) \circ (x_1, \dots, x_n, y_1, \dots, y_n, z) \\ = (a_1 + x_1, \dots, b_n + y_n, c + z + \sum_{j=1}^n (b_j x_j - a_j y_j)). \end{aligned}$$

Let

$$\hat{e}_j = \frac{\partial}{\partial x_j} + y_j \frac{\partial}{\partial z}, \quad \hat{e}_{j'} = \frac{\partial}{\partial y_j} - x_j \frac{\partial}{\partial z},$$

$1 \leq j \leq n$ be the left-invariant vector fields on H_n , in which $x_1, \dots, x_n, y_1, \dots, y_n, z$ denote the coordinates of H_n (instead of $x^1, \dots, x^n, x^{n+1}, \dots, x^{2n}, z$ used previously).

The (contact) 1-form $\Theta \equiv dz + \sum_{j=1}^n (x_j dy_j - y_j dx_j)$ annihilates $\hat{e}'_j s$ and $\hat{e}'_{j'} s$. We observe that $dx_1, dy_1, dx_2, dy_2, \dots, dx_n, dy_n, \Theta$ are dual to $\hat{e}_1, \hat{e}_{1'}, \hat{e}_2, \hat{e}_{2'}, \dots, \hat{e}_n, \hat{e}_{n'}, \frac{\partial}{\partial z}$. Define a nonnegative inner product $\langle \cdot, \cdot \rangle_{H_n}$ or $\langle \cdot, \cdot \rangle^*$ by

$$(7.5) \quad \begin{aligned} \langle dx_j, dx_k \rangle^* &= \delta_{jk}, \quad \langle dy_j, dy_k \rangle^* = \delta_{jk}, \quad \langle dx_j, dy_k \rangle^* = 0, \\ \langle \Theta, dx_j \rangle^* &= \langle \Theta, dy_k \rangle^* = \langle \Theta, \Theta \rangle^* = 0. \end{aligned}$$

We can extend the definition of the above nonnegative inner product to the situation of a general pseudohermitian manifold. Take $e_j, e_{j'} = J e_j, j = 1, 2, \dots, n$ to be an orthonormal basis in the kernel of the contact form Θ with respect to the Levi metric $\frac{1}{2}d\Theta(\cdot, J\cdot)$. Let T be the Reeb vector field of Θ (such that $\Theta(T) = 1$ and $d\Theta(T, \cdot) = 0$). Denote the dual coframe of $e_j, e_{j'}, T$ by $\theta^j, \theta^{j'}$ (and Θ). Now we can replace dx_j, dy_j by $\theta^j, \theta^{j'}$ in (7.5) to define a nonnegative inner product on a general pseudohermitian manifold:

$$(7.6) \quad \begin{aligned} \langle \theta^j, \theta^k \rangle^* &= \delta_{jk}, \quad \langle \theta^{j'}, \theta^{k'} \rangle^* = \delta_{jk}, \quad \langle \theta^j, \theta^{k'} \rangle^* = 0, \\ \langle \Theta, \theta^j \rangle^* &= \langle \Theta, \theta^{k'} \rangle^* = \langle \Theta, \Theta \rangle^* = 0. \end{aligned}$$

Define the bundle morphism $G : T^*M \rightarrow TM$ by

$$(7.7) \quad \omega(G(\eta)) = \langle \omega, \eta \rangle^*$$

for $\omega, \eta \in T^*M$. In the Riemannian case, G is in fact an isometry. In the pseudohermitian case, $G(T^*M)$ is the contact subbundle ξ of TM , the kernel of Θ . By letting $\eta = \Theta$ in (7.7), we get $G(T^*M) \subset \xi$. On the other hand, it is easy to see that $G(\theta^j) = e_j, G(\theta^{j'}) = e_{j'}$ (and $G(\Theta) = 0$). Since $e_j, e_{j'}, j = 1, 2, \dots, n$ span ξ , we have $\xi \subset G(T^*M)$. For a smooth function φ on M , we define the gradient $\nabla\varphi := G(d\varphi)$. In the pseudohermitian case, this $\nabla\varphi$ is nothing but the subgradient $\nabla_b\varphi := \sum_{j=1}^n \{e_j(\varphi)e_j + e_{j'}(\varphi)e_{j'}\}$.

Let $\Sigma \subset M$ be a (smooth) hypersurface in M with a defining function ϕ such that $\Sigma = \{\phi = 0\}$. Fix a background volume form dv_M , i.e., a nonvanishing $m+1$ form on M , where $m+1 = \dim M$ (hence $\dim \Sigma = m$). For a point ζ where $|d\phi|_*^2 := \langle d\phi, d\phi \rangle^* \neq 0$, for any $p \geq 0$, we define subriemannian area (or volume) element $dv_{\phi,p}$ and (generalized) mean curvature $H_{\phi,p}(\zeta)$ for the hypersurface $\{\phi = \phi(\zeta)\}$ by

$$(7.8) \quad \begin{aligned} dv_{\phi,p} &:= \frac{d\phi}{|d\phi|_*^{1-p}} \lrcorner dv_M, \\ d(dv_{\phi,p}) &= d\left(\frac{d\phi}{|d\phi|_*^{1-p}} \lrcorner dv_M\right) := H_{\phi,p} dv_M, \end{aligned}$$

respectively. Here the interior product for forms is defined so that

$$\eta \wedge (\omega \lrcorner dv_M) = \langle \eta, \omega \rangle^* dv_M.$$

The above notion of subriemannian area unifies those in Riemannian and pseudohermitian geometries. For more details (on the case $p = 0$), please see the Appendix: Generalized Heisenberg Geometry in [10]. Define unit normal ν_p to a hypersurface $\Sigma := \{\phi = c\}$ by the formula

$$(7.9) \quad \nu_p \lrcorner dv_M = dv_{\phi,p}.$$

Given area element $dv_{\phi,p}$ and unit normal ν_p , we can also define (generalized) mean curvature $H_{\Sigma,p}$ through a variational formula:

$$(7.10) \quad \delta_{f\nu_p} \int_{\Sigma} dv_{\phi,p} = \int_{\Sigma} f H_{\Sigma,p} dv_{\phi,p}$$

for $f \in C_0^\infty(\Sigma)$.

Proposition B.1. *On Σ , we have $H_{\phi,p} = H_{\Sigma,p}$.*

Proof. Let ι_μ denote the interior product with vector μ . From (7.10) we compute

$$\begin{aligned} \int_{\Sigma} f H_{\Sigma,p} dv_{\phi,p} &= \delta_{f\nu_p} \int_{\Sigma} dv_{\phi,p} = \int_{\Sigma} L_{f\nu_p}(dv_{\phi,p}) \\ &= \int_{\Sigma} (d \circ \iota_{f\nu_p} + \iota_{f\nu_p} \circ d)(dv_{\phi,p}) \\ &= \int_{\Sigma} d(f\nu_p \lrcorner dv_{\phi,p}) + \iota_{f\nu_p}(d(dv_{\phi,p})) \\ &= \oint_{\partial\Sigma} f\nu_p \lrcorner dv_{\phi,p} + \int_{\Sigma} \iota_{f\nu_p}(H_{\phi,p} dv_M) \text{ (by (7.8))} \\ &= 0 + \int_{\Sigma} f H_{\phi,p} dv_{\phi,p} \end{aligned}$$

since $f \in C_0^\infty(\Sigma)$ and by (7.9). It follows that $H_{\Sigma,p} = H_{\phi,p}$. \square

Let (M, J, Θ) be a pseudohermitian manifold of dimension $2n + 1$, considered as a subriemannian manifold. We take the background volume form $dv_M := \Theta \wedge (d\Theta)^n$. Let Σ be a (smooth) hypersurface of M . At nonsingular points (where $T\Sigma$ is transversal to the contact bundle ξ), we choose orthonormal (with respect to the Levi metric $\langle \cdot, \cdot \rangle_{Levi} := \frac{1}{2}d\Theta(\cdot, J\cdot)$) basis $e_1, e_{1'}, \dots, e_{n-1}, e_{(n-1)'}, e_n$ in $T\Sigma \cap \xi$, where $e_{j'} = Je_j$. We choose the horizontal (or Legendrian) normal $\nu = e_{n'}$ and a defining function ϕ satisfying $d\phi(\nu) > 0$ such that the p -area element $dv_\phi := \frac{d\phi}{|d\phi|_*} \lrcorner dv_M$ has the expression $\Theta \wedge e^1 \wedge e^{1'} \wedge \dots \wedge e^{n-1} \wedge e^{(n-1)'} \wedge e^n$ where $\Theta, e^1, e^{1'}, \dots, e^{n-1}, e^{(n-1)'}, e^n, e^{n'}$ are dual to $T, e_1, e_{1'}, \dots, e_{n-1}, e_{(n-1)'}, e_n, e_{n'}$. It follows that the associated mean curvature H_Σ , called horizontal mean curvature, is the trace of the second fundamental form:

$$H_\Sigma = \sum_{K=1,1',\dots,n} \langle \nabla_{e_K}^{p.h.} \nu, e_K \rangle_{Levi}$$

(see the Appendix in [8] for more details; note sign difference in [8]). Let $|X|_{Levi} := \langle X, X \rangle_{Levi}^{1/2}$ for $X \in \xi$. We can also express H_Σ in terms of ϕ as follows:

$$H_\Sigma = \operatorname{div}_b \frac{\nabla_b \phi}{|\nabla_b \phi|_{Levi}}$$

where ∇_b and div_b denote subgradient and subdivergence in (M, J, Θ) . For a graph $\Sigma := \{(x_1, \dots, x_n, y_1, \dots, y_n, u(x_1, \dots, x_n, y_1, \dots, y_n))\}$ in the Heisenberg group H_n , we take the defining function $\phi(x_1, \dots, x_n, y_1, \dots, y_n, z) = u(x_1, \dots, x_n, y_1, \dots, y_n) - z$. Then the resulting H_Σ coincides with the definition given by (1.12) with $\vec{F} := (-y_1, x_1, \dots, -y_n, x_n)$.

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